

Finite size lattice results for the two-boundary
Temperley–Lieb loop model

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Abstract

This thesis is concerned with aspects of the integrable Temperley–Lieb loop ($\text{TL}(n)$) model on a vertically infinite lattice with two non-trivial boundaries. The $\text{TL}(n)$ model is central in the field of integrable lattice models, and different values of n relate to different physical models. For instance, the point $n = 0$ relates to critical dense polymers and a corresponding logarithmic conformal field theory. The point $n = 1$ corresponds to critical bond percolation on the square lattice, and has connections with a combinatorial counting problem of alternating sign matrices and their symmetry classes. For general n , the $\text{TL}(n)$ model is closely related to the XXZ quantum spin chain and the 6-vertex model.

We construct the transfer matrix of the model, which describes the weights of all the possible configurations of one row of the lattice. When $n = 1$ the ground state eigenvector of this matrix can be interpreted as a probability distribution of the possible states of the system. Because of special properties exhibited by the transfer matrix at $n = 1$, we can show that the eigenvector is a solution of the q -deformed Knizhnik–Zamolodchikov equation, and we use this fact to explicitly calculate some of the components of the eigenvector. In addition, recursive properties of this transfer matrix allow us to compute the normalisation of the eigenvector, and show that it is the product of four Weyl characters of the symplectic group. Previous work in this area has produced results for the $\text{TL}(1)$ loop model with periodic boundary conditions, two trivial boundaries and mixed (one trivial, one non-trivial) boundaries, but until recently little progress had been made on the case with two non-trivial boundaries. This boundary condition lends itself to calculations relating to horizontal percolation, which is not possible with the other boundary conditions.

One of these calculations is a type of correlation function that can be interpreted as the density of percolation cluster crossings between the two boundaries of the lattice. It is an example of a class of parafermionic observables recently introduced in an attempt to rigorously prove conformal invariance of the scaling limit of critical two-dimensional lattice models. It also corresponds to the spin current in the Chalker–Coddington model of the quantum spin Hall effect. We derive an exact expression for this correlation function, using properties of the transfer matrix of the $\text{TL}(1)$ model, and find that it can be expressed in terms of the same symplectic characters as the normalisation.

In order to better understand these solutions, we use Sklyanin’s scheme to perform separation of variables on the symplectic character. We construct an invertible separating operator that transforms the multivariate character into a product of single variable polynomials. Analysing the asymptotics of these polynomials will lead, via the inverse transformation, to the asymptotic limit of the symplectic character, and thus to the

asymptotic limit of the ground state normalisation and correlation function of the loop model. We construct the separating operator by viewing the symplectic characters as eigenfunctions of a quantum integrable system, and also explicitly construct the factorised Hamiltonian for this system.

Declaration

This is to certify that:

- the thesis comprises only my original work towards the PhD except where indicated in the Preface,
- due acknowledgement has been made in the text to all other material used,
- the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

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Introduction

Statistical mechanics is the study of systems composed of large numbers of small particles. The aim of the study is to investigate how global, collective effects arise from small-scale local interactions between the particles. The systems in question are too large to make accurate predictions on the level of individual particles, so the focus is instead on statistical quantities. Examples include the average density of particles and correlation functions between two chosen points. These statistical quantities can be used to calculate derived quantities such as heat capacity, energy densities and magnetisation.

Global quantities of a system are usually expressed in terms of the system's temperature, as well as external parameters such as a magnetic field. As these parameters pass certain values, the global quantities can be observed to change dramatically, for instance at the boiling point of water. These values are known as critical points. The location and nature of any critical points of a system are more examples of quantities of interest.

Near a critical point, global quantities can be approximated by a power law (say, $(T - T_c)^\alpha$, where T_c is the critical temperature). The exponent α in this approximation is called a critical exponent, and indicates the strength of the divergence as the parameter passes the critical point. It is widely believed (and experimentally verified) that systems with the same dimensionality, symmetries, and range of interactions will have the same critical exponents — a property known as universality.

A well-known statistical mechanical model is for magnetisation in a bar of iron. Each individual atom can be thought of as a small magnet, whose alignment affects the alignment of its neighbours. The strength of the effect depends on various factors, including the temperature of the bar and the external magnetic field. At low overall temperatures the atoms will prefer to keep their current alignment, but they will also try to align with their neighbours and the external field. This leads to the phenomenon of the bar having a spontaneous magnetisation if the external field is turned off. At high temperatures the particles are more likely to lose their alignment, so there is no spontaneous magnetisation. The temperature at which the spontaneous magnetisation disappears is the critical temperature, and much interesting physics happens near this point.

The local interactions of a system can be described by an energy function E . This can be variously interpreted as a function, a matrix, or more generally an operator. By

acting on a possible state s with the energy function, we get the energy $E(s)$ associated with that state. When the system is in thermodynamic equilibrium at a temperature T , the probability of observing a state decreases as the energy of that state increases. More precisely, the probability is

$$P(s) = \frac{1}{Z} e^{-\frac{E(s)}{kT}},$$

where k is Boltzmann's constant (approx. $1.38 \times 10^{-23} \text{ J/K}$), and the factor $e^{-E(s)/kT}$ is known as the Boltzmann weight of the state s . The normalisation Z is defined as

$$Z = \sum_s e^{-\frac{E(s)}{kT}},$$

and is known as the partition function (for a derivation of these expressions see [87]). The partition function holds all the information about the probabilities of all possible states, and as such is an important quantity, especially for calculating expected values of observable properties. The partition function is also closely related to the thermodynamic free energy F of the system, by

$$F = -kT \ln Z, \quad Z = e^{-\frac{F}{kT}}.$$

The free energy is the amount of energy in the system that is available to do work.

The expected value of an observable A is the weighted sum over the values of A for all possible states, weighted by the probability of each state. In short,

$$\langle A \rangle = \frac{1}{Z} \sum_s A(s) e^{-\frac{E(s)}{kT}}.$$

Another important physical quantity is the correlation function, which describes how two spatially separated points affect each other's behaviour. For a quantity A , the correlation function is given by

$$\langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle,$$

where A_1 is the quantity evaluated at the first point, and A_2 is evaluated at the second.

A system is said to be solvable if it is possible to analytically calculate the physical quantities described above. In other words, the partition function (or equivalently, the free energy) must be calculable [4]. For physical systems, this is usually an impossible task, and as such, most of the work done in this field is either based on numerical approximations or simplified mathematical models. Nevertheless, these models can be quite powerful, as universality predicts that a model with the same dimensionality, symmetry and range of interactions as a physical system will also have exactly the same the critical exponents.

Lattice models

Some of the most commonly used models in statistical mechanics are lattice models, where the particles are located at regular intervals, and interactions are defined between them. Calculating the partition function thus involves solving the combinatorial problem of finding all possible lattice configurations, and then summing the probabilities for each configuration; the latter is generally the more difficult task.

Even though these models do not reflect reality in full detail, by letting the number of lattice sites tend to infinity they can be used to obtain a good idea of the behaviour of a physical system. This infinite limit is known as the thermodynamic limit, or equivalently the asymptotic limit. In addition, as previously mentioned, universality predicts that an appropriately designed lattice model will have the same critical exponents as a physical system.

One of the most useful tools in the study of lattice models is the method of transfer matrices. Roughly speaking, a transfer matrix describes all possible configurations of one row of the lattice. If the height and width of the lattice are M and L respectively, it can also be shown that in the large M limit, the partition function goes like

$$Z_{LM} \sim \Lambda_L^M,$$

where Λ_L is the largest eigenvalue of the transfer matrix (corresponding to the state with the lowest energy, or the ground state). The thermodynamic limit then corresponds to taking the limit as L goes to infinity.

The transfer matrix can be constructed from operators called R -matrices that act locally. If it can be shown that the R -matrices satisfy the Yang–Baxter equation, then it is possible to construct an infinite family of commuting transfer matrices. This infinite family is parameterised by the so-called spectral parameter. The fact that the family of transfer matrices commute implies that they are simultaneously diagonalisable and share eigenvectors, and since each member of the family has a different value of the spectral parameter, these eigenvectors cannot depend on the spectral parameter. This fact provides access to a host of methods for finding the eigenvalues and eigenvectors of the transfer matrix, and often leads to the system being solvable.

The most famous of these methods is the Bethe ansatz method. This method works by guessing a form for the eigenvectors of the transfer matrix (called the ansatz). The ansatz depends on a set of parameters, which are fixed by constraint equations resulting from the transfer matrix eigenvalue equation. The Bethe ansatz method relies on the existence of a pseudovacuum — an eigenvector of the transfer matrix for which there exists an annihilation operator.

Perhaps the most widely known lattice model is the Ising model, which is a model of a magnet, as described earlier. The Ising model was posed by Lenz in 1920, and solved for

one dimension by Ising in 1925 [45]. In two dimensions the solution for zero external field was given by Onsager in 1944 [74]. For non-zero external magnetic field, Zamolodchikov found a solution in 1989 for a field-theoretic version of the two-dimensional Ising model in the thermodynamic limit [108]. In 1992, Warnaar, Nienhuis and Seaton found a solution for a two-dimensional model in the same universality class [102].

The Ising model is typically defined on a square lattice. At each site is a molecule with a magnetic moment σ_i , which can take the value $+1$ (up) or -1 (down). These magnetic moments can also be regarded as spins. Each spin interacts only with its closest neighbours, and the interaction depends on the values of the spins and a coupling constant \mathcal{J} . The energy function of the Ising model is

$$E_I = \frac{\mathcal{J}}{2} \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i,$$

where the first sum is over all nearest neighbours, and B is the strength of the external field. When $\mathcal{J} < 0$, this system models a ferromagnet, since the energy is lower when adjacent spins are aligned. When $\mathcal{J} > 0$ the system models an antiferromagnet. In two dimensions, the model exhibits spontaneous magnetisation below a critical temperature, but in one dimension no such behaviour occurs.

Naturally this is a highly idealised model, as real magnetic moments can point in more than two directions. Several generalisations of the Ising model have since been proposed, including the Potts model and the n -vector model.

The Q -state Potts model, or simply Potts model [4, 101, 104], has scalar spins that can take one of Q values, and the interactions between nearest neighbours are given by Kronecker delta functions. This is a well-known and widely studied model in the field of two-dimensional lattice models. In its current form it was first written down by Potts in 1952 [79]. The energy function of the Potts model is

$$E_P = -\mathcal{J} \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j),$$

and the partition function is then

$$Z_P = \sum_{\sigma} e^{\frac{\mathcal{J}}{kT} \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j)},$$

where the sum is over all possible spin configurations. This can be rewritten as

$$Z_P = \sum_{\sigma} \prod_{\langle ij \rangle} (1 + u \delta(\sigma_i, \sigma_j)),$$

with $u = e^{\frac{\mathcal{J}}{kT}} - 1$.

The Potts model can be reformulated as a loop model in the following way [4]. Consider the system of spins as a planar graph \mathcal{L} where the spins sit on vertices and edges are drawn between neighbouring spins. On each edge, a bond may be drawn according to the following rules: if a pair of neighbouring vertices have different spin values, no bond is drawn; if the vertices share the same spin value, there is a choice of drawing a bond on the edge between them (with a weight of u), or leaving the edge blank (with weight 1). We refer to an arrangement of bonds on \mathcal{L} as a bond-graph. Then the expansion of each product in the partition function represents all the possible bond-graphs associated with a spin configuration. An example of this is shown in Figure 1.

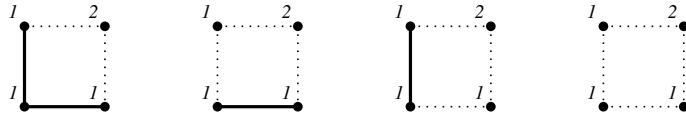


Figure 1: Possible bond-graphs for the spin configuration $((1,2),(1,1))$.

By considering instead the spin configurations possible on a particular bond-graph, we can turn the partition function into a sum over bond-graphs. Let C be the number of connected components of the bond-graph (clusters of bonds, including single vertices). All the vertices in a connected component must share a spin value. This spin value could be any of the Q , and since each connected component is independent of the others, there are Q^C ways to choose these values for the entire bond-graph. A bond-graph with l bonds has a weight of u^l . Then the partition function is a sum over all the possible bond-graphs G on \mathcal{L} ,

$$Z_P = \sum_G Q^C u^l.$$

The connection to the loop model is made by drawing loops around the bonds in the following two ways:



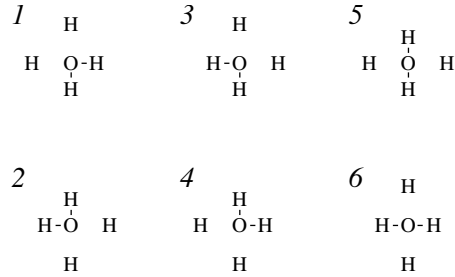
Every cluster has Q possible spin values, which leads to every closed loop in the loop expansion having a weight of \sqrt{Q} .

At $Q = 1$ the loop model is equivalent to the two-dimensional square lattice bond percolation model [77, 90]. In this model, bonds are drawn between neighbouring sites of the lattice with probability p . On an infinite lattice, percolation occurs when there is an infinite sized cluster of bonds. This model has a second order phase transition (that is, the first derivative of the free energy is continuous but the second is not) at critical probability $p = 1/2$, above which percolation occurs with probability 1.

By orienting the loops, the Potts loop model can be generalised to a directed loop model, which can itself be mapped to the six vertex model. These mappings are explained

in more detail in Chapter 12 of [4], and in [5].

The six vertex model is another classical two-dimensional lattice model, also known as the square ice model. On a square lattice, oxygen atoms are placed at each vertex, and on the edges between them are placed hydrogen atoms. A single water molecule consists of one oxygen bonded with two hydrogens, and correspondingly the ‘ice rule’ for the lattice model states that of the four hydrogens surrounding an oxygen, two will be near the oxygen and two will be distant from it. The ice rule leads to six possible configurations around a vertex:



The six possibilities are usually drawn as arrow configurations, where each arrow corresponds to a hydrogen atom and points to the nearest oxygen. Each vertex configuration

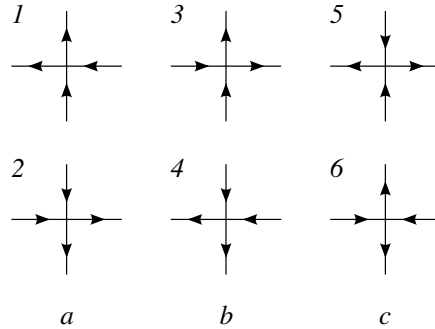


Figure 2: The 6 vertex configurations with associated Boltzmann weights a , b , and c .

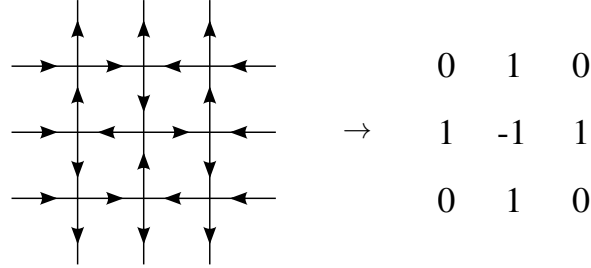
has a Boltzmann weight associated with it as shown in Figure 2. The Boltzmann weights a , b , and c appear in the partition function of the model,

$$Z_{6v} = \sum_{\text{conf.}} a^{m_a} b^{m_b} c^{m_c},$$

where the sum is over all possible configurations of the lattice and m_a , m_b and m_c are the numbers of vertices of types 1 & 2, 3 & 4, and 5 & 6 respectively.

The six vertex model has strong connections to combinatorics. When defined on a finite $M \times M$ grid, with ‘domain wall’ boundary conditions — outward-pointing arrows at the top and bottom, and inward-pointing arrows at the sides — the possible configurations

of the lattice are in bijection with $M \times M$ alternating sign matrices (ASMs). The bijection is made by replacing all vertices of type 1, 2, 3, and 4 with a 0, vertices of type 5 with a -1 and type 6 with a 1.



A similar bijection can also be made with fully packed loop diagrams (FPLs). For a thorough review see [80].

The six vertex model was solved (that is, the partition function was calculated) in 1967 by Lieb [60, 61] and Sutherland [98]. Lieb also found that the transfer matrix of the six vertex model shares eigenvectors with the Hamiltonian of the one-dimensional quantum XXZ spin chain, which was solved by Yang and Yang using the Bethe ansatz in the previous year [105, 106].

The XXZ model is a quantum generalisation of the Ising model. Its Hamiltonian is a sum over nearest neighbours,

$$H = \mathcal{J} \sum_{\langle ij \rangle} \left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \Delta \sigma_i^z \sigma_j^z \right),$$

where σ^x , σ^y , and σ^z are spin operators represented by the Pauli matrices, and Δ is known as the anisotropy parameter. When $\Delta = 1$, the system becomes the Heisenberg model, and the value $\Delta = 0$ corresponds to the free fermion point, where the model is related to the Ising model.

The eigenvectors of the six vertex transfer matrix only depend on the Boltzmann weights in a certain combination, which, as it turns out, exactly corresponds to the XXZ anisotropy parameter,

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

In fact, the transfer matrix of the six vertex model and the Hamiltonian of the one-dimensional quantum XXZ spin chain are related through the Taylor expansion of the transfer matrix,

$$T_{6v}(\mu) = T_{6v}(0) (1 + \mu H_{XXZ} + \dots),$$

where μ is the spectral parameter.

This relationship is an example of a common occurrence in classical two-dimensional lattice models, where the logarithmic derivative of the transfer matrix taken at $\mu =$

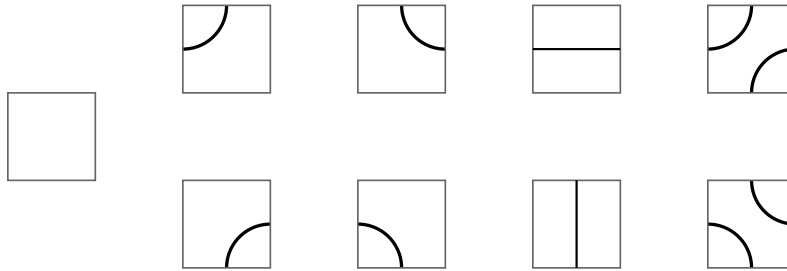
0 corresponds to the Hamiltonian of a related one-dimensional quantum system. The Hamiltonian is usually a local operator, unlike the transfer matrix, so it is often easier to deal with, however it can also contain less information.

Another generalisation of the Ising model is the n -vector model (also known as the $O(n)$ model), introduced by Stanley in 1968 [97]. In this model, each spin is a unit vector with n components. The energy function of the model is

$$E_{O(n)} = -\mathcal{J} \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j.$$

The name $O(n)$ refers to the fact that the spins have the symmetry of the orthogonal group. The limit $n \rightarrow 0$ is related to the theory of self-avoiding walks, which has an application in the study of polymers.

The n -vector model, similarly to the Potts model, has a reformulation as a loop model (see [7, 8, 35, 55, 68]), and this allows for a generalisation such that n need not be a positive integer. The $O(n)$ loop model on a square lattice has nine possible states for each lattice face:



The loops decorating the faces (or plaquettes) correspond to interactions between spins (the spins are located on the centres of the edges). Each plaquette has a weight associated with it, and various results have been obtained for systems with different restrictions on the weights.

The completely packed version [66], where the weights of all but the last two pictures are set to zero, is the one this thesis will be focusing on. The model has a rich underlying mathematical structure, best understood in terms of the Temperley–Lieb algebra. For this reason we will refer to the completely packed $O(n)$ loop model as the Temperley–Lieb loop model, or the $TL(n)$ model. This model is related to the Q -state Potts model on the square lattice by $\sqrt{Q} = n$.

The Temperley–Lieb loop model

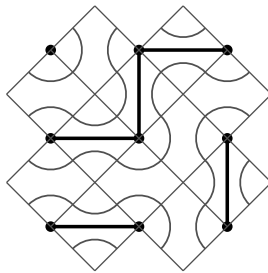
It would seem natural to define the Temperley–Lieb loop model in terms of its origins as the n -vector model. However the connection is mathematically involved [8, 33, 35, 55] and not altogether intuitive. Further, in recent years the $TL(n)$ model has garnered a

significant amount of interest quite separate from the spin model formulation. For these reasons we will define it directly as a loop model.

The $TL(n)$ loop model is defined on a horizontally finite, vertically semi-infinite square lattice, where each face of the lattice has loops drawn on it in one of two orientations, as described earlier. We choose the infinite direction to be downwards, but this is simply a convention. Weights are then given to closed loops, and we are interested in the remaining connectivities at the top of the lattice. The lattice may be wrapped around a cylinder to form the periodic $TL(n)$ model, or placed on a finite width strip with various types of boundary conditions at the left and right. If loops are allowed to end on a boundary, we call the boundary open (or non-diagonal), otherwise the boundary is closed (or diagonal). In addition to the periodic model, there are three notable versions of the model; namely the zero-boundary model with closed boundaries at the left and right, the one-boundary model with one closed and one open boundary, and the two-boundary model with two open boundaries. We are interested in the latter case.

The transfer matrix of the $TL(n)$ model describes all the possible configurations on one row of the semi-infinite lattice, and one can add rows to the lattice by acting with the transfer matrix at the top of the semi-infinite strip. When $n = 1$ the largest eigenvalue of the transfer matrix is 1, so the partition function is trivial.* These properties allow us to calculate other statistical quantities of interest for the two-boundary model with $n = 1$, such as the probability of observing a certain configuration at the top of the lattice, and a correlation function between two spatially separated points.

The $TL(n)$ model, at special values of n , is closely related to a number of different statistical mechanical models. At $n = 1$ there is an easy mapping to the two-dimensional bond percolation problem, which is made by drawing hulls around the percolation clusters as pictured:



Percolation of the bonds then corresponds to percolation of the hulls, or loops. Also at $n = 1$, the action of the transfer matrix on the lattice of the $TL(1)$ model can be interpreted as the evolution of a stochastic process (the stochastic Raise and Peel model [78]), and the ground state eigenvector of the transfer matrix is the steady state of this process. In addition, the $TL(n)$ model is closely related to the quantum XXZ spin

*This fact is in agreement with the triviality of the Potts model at $Q = 1$, when every spin has the same value. There exists a supersymmetry interpretation of this point, see [40].

chain in one dimension, as the Hamiltonians of the two models correspond to different representations of the underlying Temperley–Lieb algebra. The anisotropy parameter Δ in the XXZ model is related to n by $n = -2\Delta$.

The ground state of the TL(1) loop model with periodic boundary conditions has also been connected with the combinatorial counting problems of alternating sign matrices, fully packed loops and plane partitions. This connection has been known as the Razumov–Stroganov (RS) conjecture [30, 82–85], but was recently proved using combinatorial means by Cantini and Sportiello in [11]. The original conjecture also sparked a host of related conjectures, largely concerning different boundary conditions or topologies on the TL(1) model and different boundary conditions or symmetry requirements on the counting problems, which at the time of writing remain unproved [3, 17, 67]. Notable by its absence is a conjecture relating to the ground state of the TL(1) loop model with two distinct boundaries, which as yet has no connection to any similar combinatorial problem, though if the boundaries are identified a connection can be made [24].

A key element in the solutions of TL(n) model with different boundary conditions is the q -deformed Knizhnik–Zamolodchikov (q KZ) equation [10, 23, 26–28, 30, 48, 49, 75, 109]. With the relationship $n = -(q + q^{-1})$, it can be shown that the ground state of the TL(1) model satisfies the q KZ equation for $q = e^{2\pi i/3}$. The version of the q KZ equation used depends on the boundary conditions. This connection provides a tool set for solving for the ground state, which would otherwise be inaccessible.

Recently, a further connection has been made between the TL(1) loop model and the quantum spin Hall effect [12, 43, 52]. In particular the 2-boundary version is closely connected to this effect, since a type of correlation function between the two boundaries (see Chapter 3) corresponds to the spin current in a model of the quantum spin Hall effect.

Layout of the thesis

This thesis is concerned with properties of the TL(n) loop model with two open boundaries, at the special point $n = 1$. Each horizontal position i on the lattice has a variable z_i associated with it, and the left and right boundaries have associated variables ζ_0 and ζ_L respectively. In addition, the transfer matrix T depends on the spectral parameter w , and since it can be shown that $[T(w), T(u)] = 0$, the eigenvectors do not depend on this parameter.

Chapter 1 defines the TL(n) loop model with general parameters, first by defining the Temperley–Lieb algebra and then describing the lattice model. The transfer matrix and Hamiltonian are defined, and the specialisation to $n = 1$ is described.

The main results of Chapter 2 are the exact calculation of key components of the ground state eigenvector of the 2-boundary transfer matrix for finite system size L , as

well as the normalisation of the eigenvector. These quantities are expressed in terms of the polynomial character of the symplectic group, which is the Schur polynomial for the root system of type C . The reason why the remaining components of the eigenvector become harder and harder to specify as the system size increases is discussed.

Another quantity that is exactly calculated for finite system size, in Chapter 3, is the boundary-to-boundary correlation function, which is equivalent to the average density of percolating clusters between two points in a two-dimensional percolation model, defined on an infinite lattice of finite width. This quantity can also be interpreted as the spin current in the Chalker–Coddington model of the quantum spin Hall effect. The correlation function is expressed in terms of the same symplectic characters as the normalisation.

A key aspect of any statistical mechanical model is the thermodynamic limit. The results in Chapter 2 and Chapter 3 are obtained exactly for finite system sizes, which is usually not possible for critical systems. Generally it is only possible to derive approximate results in the thermodynamic limit. Our calculations thus allow for the calculation of exact thermodynamic quantities, by taking the limit as $L \rightarrow \infty$. It will then be possible to compare our results to the previously calculated non-rigorous continuum limit [12].

In order to obtain the thermodynamic limit, we need to consider the large L asymptotics of the symplectic character that appears in all the key results. In particular we are interested in the effect the two boundaries have in the asymptotic limit. It is therefore desirable to analyse the asymptotics of the symplectic character when all its arguments are set to 1, except for the two variables that correspond to the boundaries. In Chapter 4, a separation of variables method initiated by Sklyanin [92] is extended to the symplectic character. It is hoped that an asymptotic approximation for the separated polynomial will be easier to find than the symplectic character itself, and by reversing the process of separation the result can be used to calculate the thermodynamic limit of the physical quantities associated with the $TL(n)$ model. At present, however, this is an open problem.

Chapter 1

The Temperley–Lieb loop model

Sklyanin’s transfer matrix [92] of the $TL(n)$ model, and consequently the Hamiltonian, can be expressed in terms of algebraic generators satisfying a Temperley–Lieb algebra [100], see for example [27, 77]. The imposed boundary conditions decide which particular version of the Temperley–Lieb (TL) algebra is needed. In this chapter we will introduce the $TL(n)$ model on a strip with open boundaries on both sides, which can be described in terms of the two-boundary Temperley–Lieb (2BTL) algebra [18]. Models with two reflecting or diagonal boundaries, as well as with mixed boundaries were studied in [23, 27, 109]. The periodic version was considered in [66, 82, 84].

1.1 The Temperley–Lieb algebra

The Temperley–Lieb algebra is an sl_2 quotient of the Hecke algebra (see Appendix A). The different types of the Hecke algebra lead to different boundary conditions of the TL algebra. The 2BTL algebra comes from the Hecke algebra of type BC [18, 37], see also [16].

One of two distinguished representations of the 2BTL algebra [18] is in a space of connectivities or link patterns, described in Section 1.1.2. This representation is relevant for the $TL(n)$ model with open boundaries [46, 67], and we will use it to illustrate the generators and relations that define the TL algebras.

We will give brief definitions of the TL algebra with the four most common boundary conditions, before describing the two-boundary version in more detail. We then use the 2BTL algebra to define the $TL(n)$ loop model for general n , and in Section 1.3 discuss the specialisation $n = 1$ at which we obtain physical results for the system.

Trivial boundaries

Definition 1.1.1. The best known version of the TL algebra is generated by elements e_i , $1 \leq i \leq L-1$, which satisfy relations

$$\begin{aligned} e_i^2 &= -2 \cos \gamma e_i, & \forall i, \\ e_i e_{i+1} e_i &= e_i, & 1 \leq i \leq L-2, \\ e_i e_{i-1} e_i &= e_i, & 2 \leq i \leq L-1, \\ e_i e_j &= e_j e_i, & |i-j| > 1, \end{aligned}$$

where L is the system size and γ is a complex parameter, related to the parameter n by $n = -2 \cos \gamma$.

In the link pattern representation, the generators look like strings across a strip,

$$e_i = \begin{array}{c} \overline{\quad} \begin{array}{ccccccc} l & & \dots & i & i+1 & \dots & L \end{array} \\ \left| \begin{array}{ccccccc} \dots & & & \text{arc} & & & \end{array} \right. \end{array},$$

and multiplication in the algebra from the left corresponds to vertical concatenation of the pictures from the top. The relations can be put in terms of two rules: firstly, from the rule $e_i^2 = -2 \cos \gamma e_i$, closed loops are removed and replaced with a factor of $-2 \cos \gamma$,

$$\begin{array}{c} \overline{\quad} \begin{array}{ccccccc} l & & \dots & i & i+1 & \dots & L \end{array} \\ \left| \begin{array}{ccccccc} \dots & & & \text{loop} & & & \end{array} \right. \end{array} = -2 \cos \gamma \begin{array}{c} \overline{\quad} \begin{array}{ccccccc} l & & \dots & i & i+1 & \dots & L \end{array} \\ \left| \begin{array}{ccccccc} \dots & & & \text{arc} & & & \end{array} \right. \end{array},$$

and secondly, from the rule $e_i e_{i+1} e_i = e_i$, strings are pulled tight,

$$\begin{array}{c} \overline{\quad} \begin{array}{ccccccc} l & & \dots & i-1 & i & i+1 & \dots & L \end{array} \\ \left| \begin{array}{ccccccc} \dots & & & \text{string} & & & \end{array} \right. \end{array} = \begin{array}{c} \overline{\quad} \begin{array}{ccccccc} l & & \dots & i & i+1 & \dots & L \end{array} \\ \left| \begin{array}{ccccccc} \dots & & & \text{arc} & & & \end{array} \right. \end{array}.$$

Periodic boundaries

Definition 1.1.2. The periodic version has the generators and relations from Definition 1.1.1, as well as an extra generator e_L that satisfies the relations [50, 62]

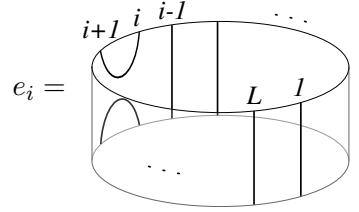
$$\begin{aligned} e_L^2 &= -2 \cos \gamma \, e_L, \\ \left. \begin{aligned} e_k e_L e_k &= e_k, \\ e_L e_k e_L &= e_L, \end{aligned} \right\} & k = 1, L-1, \\ e_L e_i &= e_i e_L, & 2 \leq i \leq L-1. \end{aligned}$$

This algebra is infinite-dimensional, so when L is even an additional relation can be imposed,

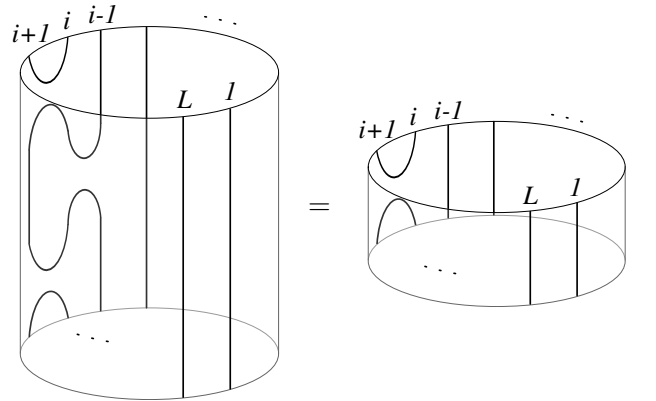
$$(e_1 e_3 \dots e_{L-1}) (e_2 e_4 \dots e_L) (e_1 e_3 \dots e_{L-1}) = \eta (e_1 e_3 \dots e_{L-1}). \quad (1.1.1)$$

This relation produces a finite-dimensional quotient of the periodic TL algebra.

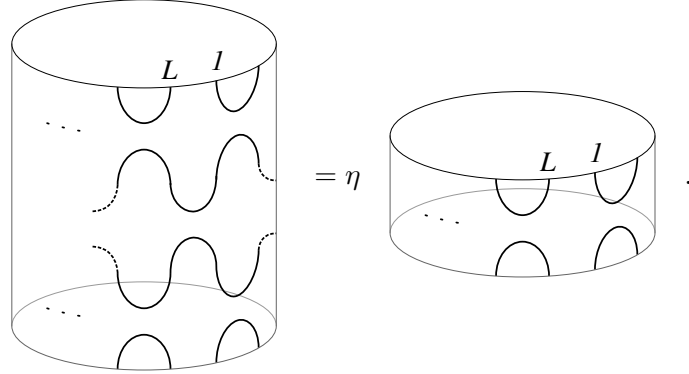
The pictorial representation is similar to the trivial boundary version, but this time the strip closes on itself to form the surface of a cylinder,



In this representation, the relation $e_i e_{i-1} e_i = e_i$ looks like



The quotient relation (1.1.1) describes the weight given to non-contractible loops,



One boundary

Definition 1.1.3. The one-boundary Temperley–Lieb (1BTL) algebra, otherwise known as the blob algebra [62], also has the generators and relations of the trivial boundary version Definition 1.1.1, along with a left boundary generator e_0 , depicted as

$$e_0 = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ \text{---} \end{array},$$

which satisfies

$$e_1 e_0 e_1 = e_0, \quad e_0^2 = \frac{-\sin \omega_0}{\sin(\omega_0 + \gamma)} e_0,$$

where ω_0 is an additional complex parameter. Note that we do not impose $e_0 e_1 e_0 = e_0$.

Pictorially the first relation is

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ \text{---} \end{array},$$

and the second is

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ \text{---} \end{array} = \frac{-\sin \omega_0}{\sin(\omega_0 + \gamma)} \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ \text{---} \end{array}.$$

Both of these relations involve a closed loop at the left boundary. However, only the second relation produces a non-trivial factor. To make clear the distinction, we introduce

the notion of parity by supposing the generators are shaded in the following way:

$$e_i = \begin{array}{c} \text{shaded semi-circle} \\ \text{unshaded semi-circle} \end{array}, \quad i \text{ even}, \quad e_i = \begin{array}{c} \text{unshaded semi-circle} \\ \text{shaded semi-circle} \end{array}, \quad i \text{ odd},$$

and e_0 has the same shading as the even case. Then a loop that has both ends connected to the left boundary produces a non-trivial factor iff the inside of the loop is shaded.

Two boundaries

Finally, the version we will be working with is the 2BTL algebra [34], which has the generators of the one-boundary case above, as well as a right boundary generator e_L depicted as

$$e_L = \left[\begin{array}{c|c|c|c|c} I & & \dots & & L \\ \hline & & & & \end{array} \right],$$

which satisfies similar relations to e_0 ,

$$e_{L-1}e_L e_{L-1} = e_L, \quad e_L^2 = \frac{-\sin \omega_L}{\sin(\omega_L + \gamma)} e_L.$$

Definition 1.1.4. Written all together, the algebraic relations for the 2BTL are

$$\begin{aligned} e_i^2 &= -2 \cos \gamma \, e_i, & 1 \leq i \leq L-1, \\ e_i^2 &= \frac{-\sin \omega_i}{\sin(\omega_i + \gamma)} e_i, & i = 0, L, \\ e_i e_{i\pm 1} e_i &= e_i, & 1 \leq i \leq L-1, \end{aligned} \tag{1.1.2}$$

along with the idempotent relations (1.1.4) described in the next section.

Loops connected to the left boundary behave as in the 1BTL, but at the right boundary the parity of L must also be taken into account. If L is even, a loop that has both ends connected to the right boundary produces a non-trivial factor iff the inside of the loop is shaded, but if L is odd the factor appears iff the inside of the loop is *not* shaded.

1.1.1 The idempotent relations

The versions of the TL algebra with 0 and 1 boundaries are both finite-dimensional. However, the periodic and 2-boundary versions are not, since it is possible to form a string winding all the way around the cylinder (for the periodic case) or stretching from the left boundary to the right (for the two-boundary case), as in Figure 1.1.

It was shown in [18] that all finite-dimensional irreducible representations of the 2BTL satisfy two additional relations, which we will describe now.

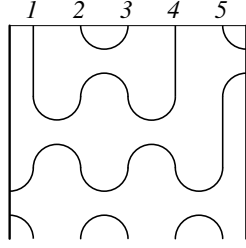


Figure 1.1: The arrangement $e_2e_5e_1e_3e_0e_2e_4$ when $L = 5$ has a loop stretching from the left boundary to the right boundary.

Definition 1.1.5. We first define two (unnormalised) idempotents I_1 and I_2 as follows:

$$I_1 = \begin{cases} e_1e_3 \cdots e_{L-1}, & L \text{ even,} \\ e_1e_3 \cdots e_L, & L \text{ odd,} \end{cases} \quad I_2 = \begin{cases} e_0e_2 \cdots e_L, & L \text{ even,} \\ e_0e_2 \cdots e_{L-1}, & L \text{ odd.} \end{cases} \quad (1.1.3)$$

For example, when $L = 6$ the idempotents are

$$I_1 = \left[\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \text{Diagram with arcs connecting (1,2), (3,4), (5,6) and (2,3), (4,5)} \end{array} \right], \quad I_2 = \left[\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \text{Diagram with arcs connecting (1,2), (3,4), (5,6) and (2,3), (4,5)} \end{array} \right].$$

The double quotient of the 2BTL algebra has the additional relations:

$$I_1I_2I_1 = bI_1, \quad I_2I_1I_2 = bI_2, \quad (1.1.4)$$

where b is an additional parameter. In our pictorial representation, these relations have the effect of removing pairs of loops stretching from the left boundary to the right and replacing them with a factor of b .

1.1.2 Link pattern space

Here we will define the space LP_L spanned by link patterns (sometimes called connectivities) in terms of the loop representation of the 2BTL generators above. This space is equivalent to the space spanned by (a variant of) anchored cross paths [81]. The space LP_L forms the state space of the $\text{TL}(n)$ loop model [91]. An example of a link pattern is given in Figure 1.2.

Definition 1.1.6. A link pattern is a non-crossing matching of the integers $0, 1, \dots, L+1$. The matching between the integers $1, \dots, L$ is pairwise, whereas 0 and $L+1$ may be matched with, or connected to, an arbitrary number of other integers. The integers 0 and $L+1$ are respectively referred to as the left and right boundary.

We can express elements of LP_L as words in the generators of the 2BTL. We choose one of the idempotents to be the shortest word, and act with any combination of the 2BTL generators. Then by the relations (1.1.2) and (1.1.4), the resulting picture will

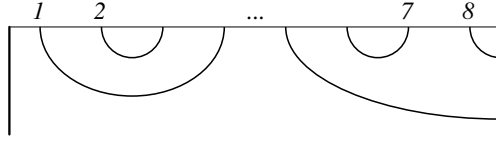
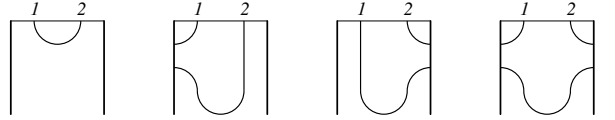


Figure 1.2: A link pattern for $L = 8$.

reduce to one of the link patterns of size L , multiplied by the weight as introduced by the relations. For example, with $L = 2$ we define the shortest word to be $I_1 = e_1$, and acting with the combination $e_0 e_1$ produces

$$e_0 e_1 I_1 = -2 \cos \gamma \, e_0 I_1.$$

In fact, the four basis elements of LP_2 are I_1 , $e_0 I_1$, $e_2 I_1$ and $e_0 e_2 I_1$, and these are represented respectively by the pictures



Note the line connecting the two boundaries in the final picture. If we had defined the shortest word to be $I_2 = e_0 e_2$ instead of I_1 , a line would appear on the first three pictures and not the final one. Using the word representation of link patterns and starting from I_1 , the link pattern with 1 connected to the left boundary and 2 to the right will always have one of these lines, and the others never will. Because of this, and because it is always possible to find out whether a picture should have a line, we omit single lines connecting the two boundaries when referring to a link pattern. In this way, the above pictures can be represented as the link patterns in Figure 1.3.

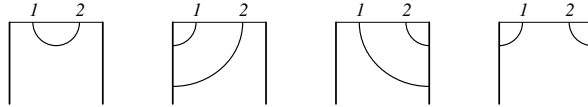


Figure 1.3: The link patterns for $L = 2$, corresponding to I_1 , $e_0 I_1$, $e_2 I_1$ and $e_0 e_2 I_1$ respectively.

We use a shorthand notation for the link patterns, in terms of a sequence of opening ‘(’ and closing ‘)’ parentheses. The i th bracket in the sequence refers to whether the i th site is connected to some place to the right of it (opening bracket) or to the left (closing bracket). Since the loops are non-crossing and every site from 1 to L must be connected, this provides a unique labelling. The above link patterns for $L = 2$ are thus

given respectively by

$$(), \quad), \quad ((, \quad)(.$$

Because we can independently place an opening or closing parenthesis at each site, the dimension of the space LP_L of link patterns of size L for the two-boundary Temperley–Lieb algebra is

$$\dim \text{LP}_L = 2^L.$$

1.1.3 Path representation

We now present a representation of the link pattern space using the graphical depiction of e_i as a tilted square tile decorated with small loop segments; and of e_0 and e_L as similarly decorated half-tiles lying against the boundaries of the picture, as shown in Figure 1.4.

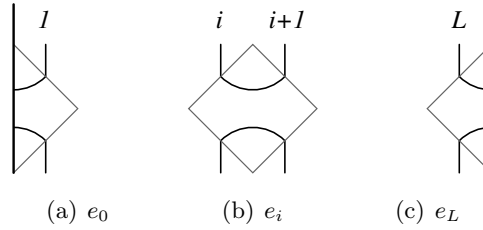


Figure 1.4: Tiles for the 2BTL generators.

The shortest word, given by one of the two idempotents, is depicted as a row of tilted half-tiles, which lie along the bottom of the picture, as in Figure 1.5.

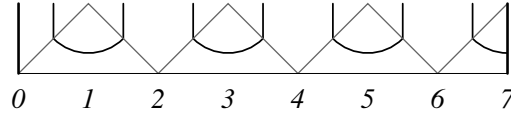
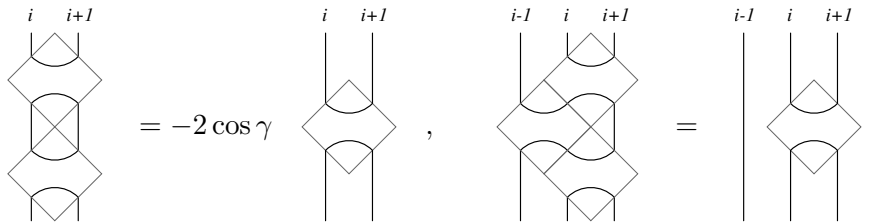


Figure 1.5: The idempotent I_1 for $L = 7$. The label i on the x-axis refers to the centre of the tile e_i .

Multiplication in the 2BTL algebra corresponds to vertical concatenation of the tiles. As an example, below are shown the algebraic relations $e_i^2 = -2 \cos \gamma e_i$ and $e_i e_{i-1} e_i = e_i$. The other relations are similar.



Because of these relations, each link pattern corresponds uniquely to a path, which is traced out by the top of the tiles. A path is defined in this representation as beginning at the left boundary, ending at the right, and taking one step either up or down for each step to the right. The shorthand notation defined in the previous section works with the path representation as well, where steps down and up are symbolised by ‘)’ and ‘(’ respectively. It is easily seen (as in Figure 1.6) that when the path steps up, the loop in the link pattern connects somewhere to the right, and the opposite is true for down steps.

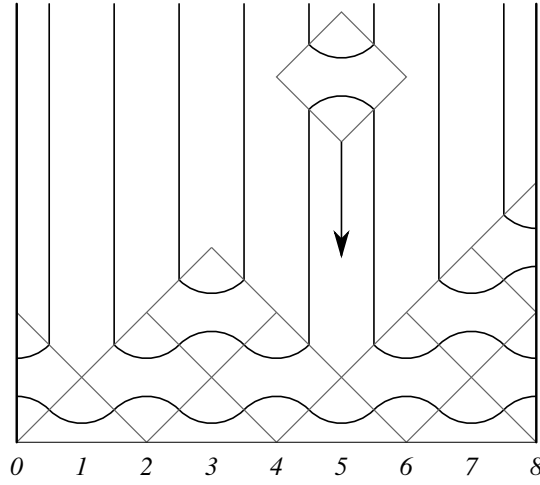


Figure 1.6: Path representation of the action of e_5 on the link pattern $)())(($. The result will be the link pattern $)())(($.

1.1.4 Spin chain representation

The 2BTL algebra has another distinguished representation, also of dimension 2^L [67]. This representation is in the tensor product space $(\mathbb{C}^2)^{\otimes L}$, giving rise to the quantum XXZ spin chain with non-diagonal boundary conditions on both sides. We define the Pauli spin matrices,

$$\begin{aligned}\sigma^x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \sigma^y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \\ \sigma^z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\end{aligned}$$

and will use the notation

$$\sigma_i^{x,y,z} = 1^{\otimes i-1} \otimes \sigma^{x,y,z} \otimes 1^{\otimes L-i}.$$

In the spin chain representation the Temperley–Lieb generators take the form [19]

$$\begin{aligned} e_i &\mapsto \frac{1}{2} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos \gamma \left(\sigma_i^z \sigma_{i+1}^z - 1 \right) - i \sin \gamma \left(\sigma_i^z - \sigma_{i+1}^z \right) \right), \\ e_0 &\mapsto \frac{1}{2 \sin(\omega_0 + \gamma)} \left(\cos \theta_1 \sigma_1^x + \sin \theta_1 \sigma_1^y + i \cos \omega_0 \sigma_1^z - \sin \omega_0 \right), \\ e_L &\mapsto \frac{1}{2 \sin(\omega_L + \gamma)} \left(\cos \theta_2 \sigma_L^x + \sin \theta_2 \sigma_L^y - i \cos \omega_L \sigma_L^z - \sin \omega_L \right), \end{aligned}$$

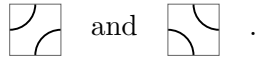
where θ_1 and θ_2 are related to the idempotent parameter b by

$$b = \begin{cases} \frac{\cos(\theta_1 - \theta_2) + \cos(\gamma + \omega_0 + \omega_L)}{2 \sin(\gamma + \omega_0) \sin(\gamma + \omega_L)}, & L \text{ even,} \\ \frac{\cos(\theta_1 - \theta_2) + \cos(\omega_0 - \omega_L)}{2 \sin(\gamma + \omega_0) \sin(\gamma + \omega_L)}, & L \text{ odd.} \end{cases}$$

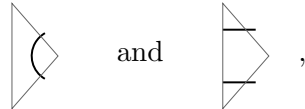
Note that b only depends on the difference of these two parameters. Due to the rotational symmetry in the spin x - y plane, the extra freedom $(\theta_1 + \theta_2)$ can be interpreted as a free gauge parameter on which the algebra does not depend.

1.2 The TL(n) lattice

The two-boundary TL(n) loop model is defined on a lattice of width L , vertically infinite with a boundary on the left and right hand sides. Each face of the lattice is decorated with loops, which can end on the boundary or close back on themselves. Each bulk (square) face has two possible configurations of loops,



The boundary (triangle) faces also have two possible configurations, one joining the loops on two adjacent rows of the lattice, the other connecting them both to the boundary. At the left boundary the possibilities are



and at the right boundary the above pictures are reflected. Because of the action of the boundary faces, it is easy to see that the rows of the lattice come in pairs, as illustrated in Figure 1.7.

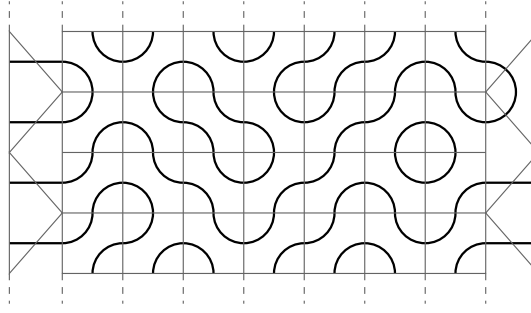


Figure 1.7: One of the configurations of four rows of the lattice for $L = 7$.

One of the important aspects of this model is how the loops connect different parts of the lattice. If we choose a configuration for each of the faces, and cut the lattice along a horizontal line between two pairs of rows, it is clear that below the line there are a collection of open loop segments, which connect different horizontal positions (see Figure 1.8). In the example, position 1 is connected to the left boundary, position 2 to 5,

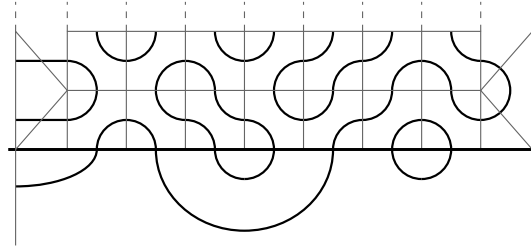
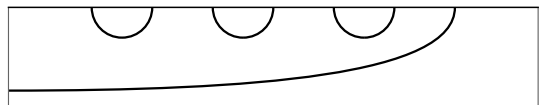


Figure 1.8: The link pattern below the line is produced by the bottom two rows of the lattice in Figure 1.7.

3 to 4, and 6 to 7. This is a clear example of a link pattern as defined in Section 1.1.2. As in the 2BTL algebra, we can ignore the specific paths the loops take and replace closed loops and loops that have both ends connected to a boundary by their respective weights. Once this is done the only thing left to consider is the remaining link pattern, which we call α . Every configuration of the semi-infinite lattice below the horizontal line has one of the 2^L link patterns at the top.

The two rows of the lattice above the link pattern can have one of 2^{2L+2} possible configurations, and each one will map a link pattern into another. As an example, the configuration in Figure 1.8 produces the link pattern



and introduces the weight $4 \cos^2 \gamma$ from the two closed loops.* We represent the two lattice rows as a matrix T (called the transfer matrix), with the ij th entry being the sum of weights produced by all the configurations that map the j th link pattern to the i th (where there is some ordering on the link patterns).

A vector with a basis in the space of link patterns can be written as

$$|\Psi\rangle = \sum_{\alpha \in \text{LP}_L} \psi_\alpha |\alpha\rangle.$$

If this vector is the unique ‘ground state’ eigenvector of T ,

$$T|\Psi\rangle = \Lambda_{\max} |\Psi\rangle,$$

where Λ_{\max} is the maximum eigenvalue, then repeated action of T on some initial state $|\text{in}\rangle$ produces

$$T^k |\text{in}\rangle \sim \Lambda_{\max}^k |\Psi\rangle, \quad \text{as } k \rightarrow \infty.$$

This process can be seen as building up a semi-infinite lattice with infinite copies of the transfer matrix. The eigenvector then expresses the relative weights of all possible link patterns of size L , produced by the configurations on the lattice.

As shown by Di Francesco and Zinn-Justin for other types of boundary conditions [27, 109], it is possible to derive exact closed form expressions for certain properties of the ground state eigenvector for finite system sizes. To achieve this one needs to generalise the model just described, by considering an inhomogeneous version of the transfer matrix. This is done in the next section.

1.2.1 Baxterisation

In order to define the transfer matrix we will first introduce the operators \check{R} and \check{K} , as well as their unchecked versions, to be defined shortly. We furthermore list some useful properties that we will need in later calculations. Throughout the following we will use the notation $[z]$ for

$$[z] = z - z^{-1},$$

and define

$$q = e^{i\gamma}.$$

Definition 1.2.1. The Baxterised elements $\check{R}_i(z)$, and the boundary Baxterised elements

*The loop at the left boundary does not contribute a weight, as it only involves one left boundary tile, and is therefore a loop produced by the rule $e_1 e_0 e_1 = e_1$. At the end of this chapter all the weights will be set to 1 and this technicality will no longer be important.

$\check{K}_0(z, \zeta)$ and $\check{K}_L(z, \zeta)$ of the Temperley–Lieb algebras are defined as

$$\begin{aligned}\check{R}_i(z) &= \frac{1}{\kappa(z)} \frac{[q/z] - [z] e_i}{[qz]}, \\ \check{K}_i(z, \zeta) &= \frac{k(z, \zeta, \omega_i) + [q e^{i\omega_i}] [z^2] e_i}{k(1/z, \zeta, \omega_i)}, \quad i = 0, L,\end{aligned}\tag{1.2.1}$$

where the parameter z is called the spectral parameter. Each boundary element can also be equipped with an additional free parameter ζ .

The weight $k(z, \zeta, \omega_i)$ here is given by

$$k(z, \zeta, \omega) = [z e^{i\omega/2} / \zeta] [z e^{i\omega/2} \zeta],\tag{1.2.2}$$

and $\kappa(z)$ is defined in terms of q -Pochhammer symbols [48],

$$\kappa(z) = \begin{cases} z \frac{(q^6 z^2; q^4)_\infty (q^4 z^{-2}; q^4)_\infty}{(q^6 z^{-2}; q^4)_\infty (q^4 z^2; q^4)_\infty}, & |q| < 1, \\ \frac{1}{z} \frac{(q^{-6} z^{-2}; q^{-4})_\infty (q^{-4} z^2; q^{-4})_\infty}{(q^{-6} z^2; q^{-4})_\infty (q^{-4} z^{-2}; q^{-4})_\infty}, & |q| > 1, \end{cases}\tag{1.2.3}$$

where

$$(a; b)_\infty = \prod_{n=0}^{\infty} (1 - ab^n).$$

This normalisation factor satisfies the functional relations

$$\begin{aligned}\kappa(z) \kappa(1/z) &= 1, \\ \frac{\kappa(-q/z)}{\kappa(z)} &= \frac{-[qz]}{[q^2/z]},\end{aligned}\tag{1.2.4}$$

and has the special value

$$\kappa(1) = 1.\tag{1.2.5}$$

Note that the definition (1.2.3) for κ is non-analytic across the unit circle $|q| = 1$. For these values of q there is an alternate definition of κ , described in [49]. At the special point $q = e^{2\pi i/3}$, we can take $\kappa \equiv 1$, as this satisfies the functional relations (1.2.4) and (1.2.5). We note, however, that this choice for κ is not the limit of either function in (1.2.3) as $q \rightarrow e^{2\pi i/3}$.

Proposition 1.2.1. *The Baxterised elements obey the usual Yang–Baxter and reflection*

equations with spectral parameters:

$$\begin{aligned}\check{R}_i(z)\check{R}_{i+1}(zw)\check{R}_i(w) &= \check{R}_{i+1}(w)\check{R}_i(zw)\check{R}_{i+1}(z), \\ \check{K}_0(z, \zeta)\check{R}_1(zw)\check{K}_0(w, \zeta)\check{R}_1(w/z) &= \check{R}_1(w/z)\check{K}_0(w, \zeta)\check{R}_1(zw)\check{K}_0(z, \zeta), \\ \check{K}_L(z, \zeta)\check{R}_{L-1}(zw)\check{K}_L(w, \zeta)\check{R}_{L-1}(w/z) &= \check{R}_{L-1}(w/z)\check{K}_L(w, \zeta)\check{R}_{L-1}(zw)\check{K}_L(z, \zeta).\end{aligned}\quad (1.2.6)$$

They furthermore satisfy the unitarity relations

$$\begin{aligned}\check{R}_i(z)\check{R}_i(1/z) &= 1, \\ \check{K}_i(z, \zeta)\check{K}_i(1/z, \zeta) &= 1, \quad i = 0, L.\end{aligned}\quad (1.2.7)$$

Proof. These relations can be easily checked using the algebraic rules (1.1.2), or using a graphical notation like the one in Figure 1.6. \square

The above Baxterised elements are special cases of R -matrices, which can be defined more generally using the Hecke algebra. This is explained in more detail in Appendix A.

We now introduce a graphical version of the Baxterised elements, using the planar Temperley–Lieb–Jones algebra [50], which we will be able to use in a more general context than Figure 1.6.

Definition 1.2.2. We define the R -operator $R(w, z)$ to be the following linear combination of pictures:

$$R(w, z) = \frac{1}{\kappa(w/z)} \left(\frac{[qz/w]}{[qw/z]} \begin{array}{|c|} \hline \text{diagram 1} \\ \hline \end{array} + \frac{[z/w]}{[qw/z]} \begin{array}{|c|} \hline \text{diagram 2} \\ \hline \end{array} \right),$$

and graphically abbreviate $R(w, z)$ by

$$R(w, z) = w \begin{array}{c} \uparrow \\ \begin{array}{|c|} \hline \text{diagram 3} \\ \hline \end{array} \\ \downarrow \\ z \end{array}.$$

Note that we can use this picture in any orientation, as the arrows uniquely determine how the spectral parameters z and w enter in R . It is worthwhile to point out that

$R(cw, cz) = R(w, z)$, for any constant c . We also define the boundary K -operators by

$$\begin{aligned}
K_0(w) &= \frac{k(qw, \zeta_0, \omega_0)}{k(1/qw, \zeta_0, \omega_0)} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) + \frac{[q e^{i\omega_0}][(qw)^2]}{k(1/qw, \zeta_0, \omega_0)} \begin{array}{c} \diagup \\ \diagdown \\ \hline \diagup \\ \diagdown \end{array} \\
&= \begin{array}{c} \text{Diagram: A diamond shape with a white circle at the top vertex. Two arrows originate from the circle: one pointing up-right labeled } w, \text{ and one pointing down-right labeled } w^{-1}. \text{ The bottom vertex is labeled } \zeta_0. \end{array}, \\
K_L(w) &= \frac{k(w, \zeta_L, \omega_L)}{k(1/w, \zeta_L, \omega_L)} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) + \frac{[q e^{i\omega_L}][w^2]}{k(1/w, \zeta_L, \omega_L)} \begin{array}{c} \diagdown \\ \diagup \\ \hline \diagdown \\ \diagup \end{array} \\
&= \begin{array}{c} \text{Diagram: A diamond shape with a black circle at the top vertex. Two arrows originate from the circle: one pointing up-left labeled } w^{-1}, \text{ and one pointing down-left labeled } w. \text{ The bottom vertex is labeled } \zeta_L. \end{array}.
\end{aligned}$$

The Baxterised elements R , K_0 and K_L will be used to define the transfer matrix of the system. We can also write K_0 as

$$\begin{array}{c} \text{Diagram: A diamond shape with a white circle at the top vertex. Two arrows originate from the circle: one pointing up-right labeled } w, \text{ and one pointing down-right labeled } w^{-1}. \text{ The bottom vertex is labeled } \zeta_0. \end{array} = \begin{array}{c} \text{Diagram: A diamond shape with a black circle at the top vertex. Two arrows originate from the circle: one pointing up-right labeled } -qw, \text{ and one pointing down-right labeled } -1/qw. \text{ The bottom vertex is labeled } \zeta_0. \end{array}, \quad (1.2.8)$$

where

$$\begin{array}{c} \text{Diagram: A diamond shape with a black circle at the top vertex. Two arrows originate from the circle: one pointing up-right labeled } w, \text{ and one pointing down-right labeled } w^{-1}. \text{ The bottom vertex is labeled } \zeta_0. \end{array} = \frac{k(w, \zeta_0, \omega_0)}{k(1/w, \zeta_0, \omega_0)} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) + \frac{[q e^{i\omega_0}][w^2]}{k(1/w, \zeta_0, \omega_0)} \begin{array}{c} \diagup \\ \diagdown \\ \hline \diagup \\ \diagdown \end{array},$$

which will be useful for defining the relations satisfied by K_0 .

Proposition 1.2.2. *The unitarity relations (1.2.7) for R and K can be graphically depicted as*

$$\begin{array}{c} \text{Diagram: Two crossing lines forming a diamond shape. The top-left line is labeled } z, \text{ the top-right line is labeled } w, \text{ the bottom-left line is labeled } w, \text{ and the bottom-right line is labeled } z. \end{array} = \begin{array}{c} \xrightarrow{z} \\ \xrightarrow{w} \end{array}, \quad (1.2.9)$$

and

$$\begin{array}{c} \text{Diagram: A vertical chain of three diamonds. The top diamond has a black circle at its top vertex with an arrow pointing up labeled } z. \text{ The middle diamond has a black circle at its top vertex with an arrow pointing down labeled } z^{-1}. \text{ The bottom diamond has a black circle at its top vertex with an arrow pointing down labeled } z. \end{array} = \begin{array}{c} \text{Diagram: A vertical chain of three diamonds. The top diamond has a black circle at its top vertex with an arrow pointing down labeled } z^{-1}. \text{ The middle diamond has a black circle at its top vertex with an arrow pointing up labeled } z. \text{ The bottom diamond has a black circle at its top vertex with an arrow pointing down labeled } z. \end{array} = \begin{array}{c} \uparrow \\ z \end{array}.$$

In addition, the Yang–Baxter and reflection equations (1.2.6) can be written as

$$\begin{array}{c} \begin{array}{c} u \\ \diagdown \\ \text{diamond} \\ \diagup \\ v \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array} = \begin{array}{c} \begin{array}{c} u \\ \text{square} \\ \text{square} \\ \text{square} \\ v \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array}, \quad (1.2.10)$$

and

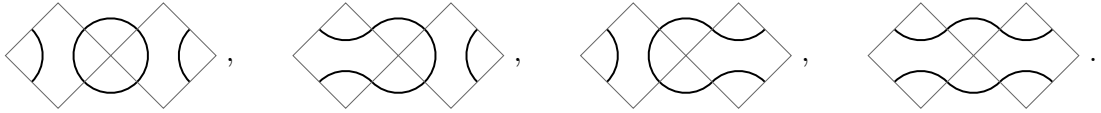
$$\begin{array}{c} \begin{array}{c} z \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ z^{-1} \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array} = \begin{array}{c} \begin{array}{c} z \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ z^{-1} \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array}, \quad \begin{array}{c} \begin{array}{c} z^{-1} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ z \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w^{-1} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array} = \begin{array}{c} \begin{array}{c} z^{-1} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ \text{diamond} \\ z \end{array} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w^{-1} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array}.$$

We furthermore note the crossing relation satisfied by R , i.e., $R(z, w) = R(-qw, z)$, which graphically reads [48]

$$\begin{array}{c} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} z \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} = -qw \begin{array}{c} \begin{array}{c} \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} z \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \begin{array}{c} w \\ \text{square} \\ \text{square} \\ \text{square} \\ \text{square} \end{array} \end{array}. \quad (1.2.11)$$

The above relations are straightforward to prove from the definitions, by considering two loops to be the same if they have the same connectivities, and replacing closed loops with a factor of $-(q + q^{-1})$. The unitarity relation for R (1.2.9) is proved here as an example.

Proof. The LHS of (1.2.9) produces four pictures,



The first three of these have the same connectivity, so they can be grouped together with coefficient

$$\frac{1}{\kappa(w/z)\kappa(z/w)} \left(\frac{-(q + q^{-1})[w/z][z/w] + [qw/z][z/w] + [w/z][qz/w]}{[qz/w][qw/z]} \right) = 0.$$

This leaves only the fourth picture, which is equal to the identity. \square

Finally, we will also define slightly different versions of K_0 and K_L , which will be

useful in proving the commutativity of the transfer matrix,

$$\begin{array}{ccc}
\begin{array}{c} \text{Diagram 1: A diamond shape with a white circle at the top vertex. An arrow labeled } w \text{ points from the top vertex to the right vertex. An arrow labeled } -l/qw \text{ points from the top vertex to the bottom vertex. The bottom vertex is labeled } \zeta_0. \end{array} & = & \begin{array}{c} \text{Diagram 2: A diamond shape with a white circle at the top vertex. An arrow labeled } w \text{ points from the top vertex to the right vertex. An arrow labeled } w^{-l} \text{ points from the top vertex to the bottom vertex. The bottom vertex is labeled } \zeta_0. \end{array} , \\
\begin{array}{c} \text{Diagram 3: A diamond shape with a black circle at the top vertex. An arrow labeled } w^{-l} \text{ points from the top vertex to the left vertex. An arrow labeled } -w/q \text{ points from the top vertex to the bottom vertex. The bottom vertex is labeled } \zeta_L. \end{array} & = & \begin{array}{c} \text{Diagram 4: A diamond shape with a black circle at the top vertex. An arrow labeled } w^{-l} \text{ points from the top vertex to the left vertex. An arrow labeled } w \text{ points from the top vertex to the bottom vertex. The bottom vertex is labeled } \zeta_L. \end{array} .
\end{array} \quad (1.2.12)$$

These K -matrices satisfy the following versions of the reflection equations,

$$\begin{array}{ccc}
\begin{array}{c} \text{Diagram 1: A vertical chain of three diamonds. The top diamond has a white circle at the top vertex. The middle diamond has a white circle at the top vertex. The bottom diamond has a white circle at the top vertex. Arrows: } z^{-l} \text{ (top left), } w^{-l} \text{ (top right), } -z/q \text{ (bottom left), } -w/q \text{ (bottom right).} \end{array} & = & \begin{array}{c} \text{Diagram 2: A vertical chain of three diamonds. The top diamond has a white circle at the top vertex. The middle diamond has a white circle at the top vertex. The bottom diamond has a white circle at the top vertex. Arrows: } z^{-l} \text{ (top left), } w^{-l} \text{ (top right), } -z/q \text{ (bottom left), } -w/q \text{ (bottom right).} \end{array} , \\
\begin{array}{c} \text{Diagram 3: A vertical chain of three diamonds. The top diamond has a black circle at the top vertex. The middle diamond has a black circle at the top vertex. The bottom diamond has a black circle at the top vertex. Arrows: } z^{-l} \text{ (top left), } w^{-l} \text{ (top right), } -z/q \text{ (bottom left), } -w/q \text{ (bottom right).} \end{array} & = & \begin{array}{c} \text{Diagram 4: A vertical chain of three diamonds. The top diamond has a black circle at the top vertex. The middle diamond has a black circle at the top vertex. The bottom diamond has a black circle at the top vertex. Arrows: } z^{-l} \text{ (top left), } w^{-l} \text{ (top right), } -z/q \text{ (bottom left), } -w/q \text{ (bottom right).} \end{array} .
\end{array} \quad (1.2.13)$$

These are easily proved using the crossing relation (1.2.11).

1.2.2 Transfer matrix

Definition 1.2.3. Using the pictorial versions of the R and K -matrices in Definition 1.2.2 we define Sklyanin's double row transfer matrix $T_L(w) = T_L(w; z_1, \dots, z_L; \zeta_0, \zeta_L)$ pictorially in the following way (see [27, 77, 92]),

$$T_L(w) = \begin{array}{c} \text{Diagram: A horizontal chain of diamonds. The left end is a white circle labeled } \zeta_0. The right end is a black circle labeled } \zeta_L. \text{ The chain consists of } L \text{ diamonds. The top row of diamonds has vertices labeled } z_1, z_2, \dots, z_{L-1}, z_L. \text{ The bottom row of diamonds has vertices labeled } z_1, z_2, \dots, z_{L-1}, z_L. \text{ Arrows: } w^{-l} \text{ (top left), } w \text{ (bottom right).} \end{array} .$$

This can also be written as

$$T_L(w) = \text{Tr}_w \left(K_0(w^{-1}) R(w, z_1) \dots R(w, z_L) K_L(w) R(z_L, w^{-1}) \dots R(z_1, w^{-1}) \right) ,$$

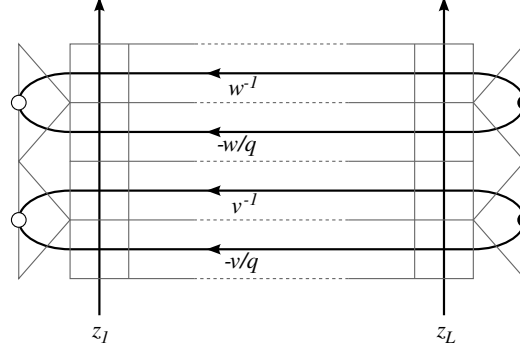
where, in terms of pictures, the trace means that we join the two ends of the line with the parameter w attached to it, as shown in the above diagram.

Proposition 1.2.3. *All the possible values of w give us a commuting family of transfer matrices (see for example [92]), i.e.,*

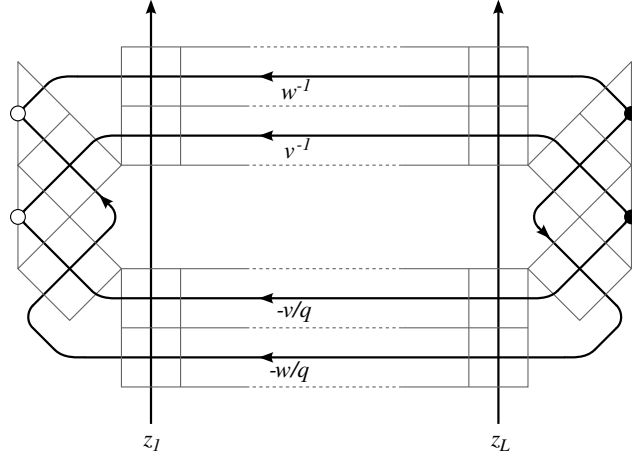
$$[T_L(v), T_L(w)] = 0,$$

and hence $T(w)$ defines an integrable lattice model.

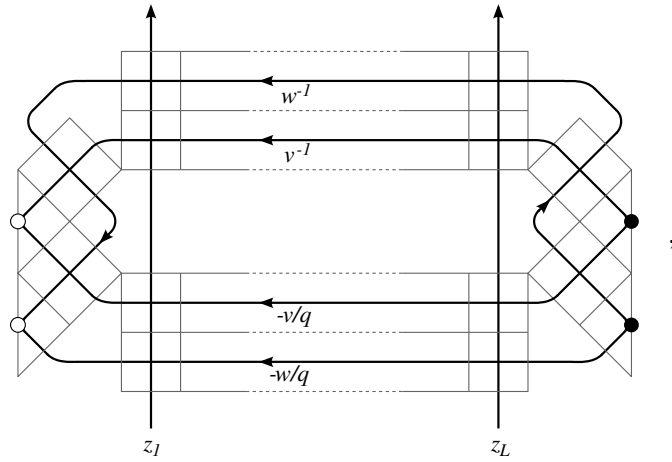
Proof. First, note that with the notation (1.2.12) and the crossing relation, $T(w)T(v)$ may be depicted



Using the unitarity relation (1.2.9) twice, and repeated application of the Yang-Baxter equation (1.2.10), the above becomes

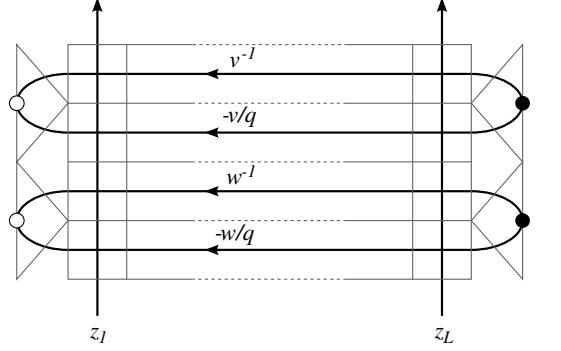


Now, at both boundaries, the reflection equations (1.2.13) can be applied, resulting in



and the process involving the Yang–Baxter equation and unitarity relations can be re-

versed, to finally produce



which is the graphical depiction of $T(v)T(w)$. \square

As a consequence of this commutativity, the eigenvectors of T_L do not depend on the spectral parameter w , but only on z_1, \dots, z_L ,

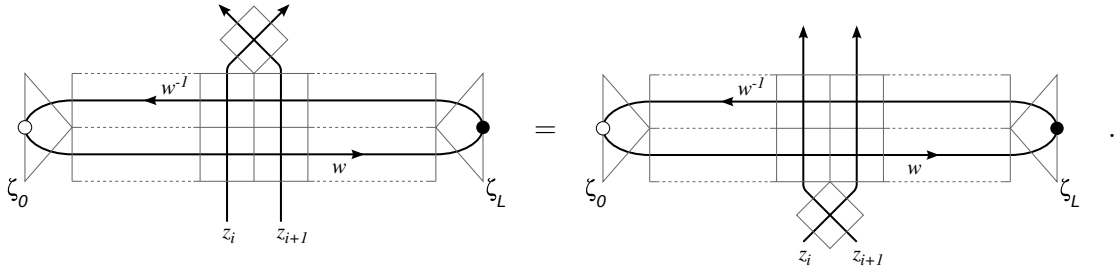
$$T_L(w; z_1, \dots, z_L) |\Psi(z_1, \dots, z_L)\rangle \propto |\Psi(z_1, \dots, z_L)\rangle.$$

Following [27, 29], we note that the Yang–Baxter and reflection equations (1.2.6) also immediately imply the following interlacing conditions of the transfer matrix with \check{R}_i , \check{K}_0 and \check{K}_L .

Proposition 1.2.4.

$$\begin{aligned} \check{R}_i(z_i/z_{i+1})T_L(w; z_1, \dots, z_L) &= T_L(w; z_1, \dots, z_{i+1}, z_i, \dots, z_L)\check{R}_i(z_i/z_{i+1}), \\ \check{K}_0(z_1^{-1}, \zeta_0)T_L(w; z_1, \dots, z_L) &= T_L(w; z_1^{-1}, z_2, \dots, z_L)\check{K}_0(z_1^{-1}, \zeta_0), \\ \check{K}_L(z_L, \zeta_L)T_L(w; z_1, \dots, z_L) &= T_L(w; z_1, \dots, z_{L-1}, z_L^{-1})\check{K}_L(z_L, \zeta_L). \end{aligned} \quad (1.2.14)$$

Pictorially, the first relation is



The other relations have similar depictions.

1.2.3 Hamiltonian

Definition 1.2.4. The Hamiltonian of the $\text{TL}(n)$ loop model with open boundaries is defined in terms of the logarithmic derivative of the transfer matrix with respect to w at the point $z_1 = \dots = z_L = w = 1$;

$$\begin{aligned} H &= \frac{[q]}{4} \left. \frac{\partial}{\partial w} \right|_{w=1} \log T(w; 1, \dots, 1) \\ &= \frac{[q]}{4} \left(\frac{\partial \check{K}_0(w)}{\partial w} + \frac{\partial \check{K}_L(w)}{\partial w} + 2 \sum_{j=1}^{L-1} \frac{\partial \check{R}_j(w)}{\partial w} \right) \Big|_{w=1} + \text{const.} \end{aligned}$$

This can be calculated and expressed as the following operator,

$$H = c_0 \left(\frac{-\sin \omega_0}{\sin(\omega_0 + \gamma)} - e_0 \right) + c_L \left(\frac{-\sin \omega_L}{\sin(\omega_L + \gamma)} - e_L \right) + \sum_{j=1}^{L-1} (-2 \cos \gamma - e_j) + \text{const.}, \quad (1.2.15)$$

where c_0, c_L are

$$c_i = \frac{4 \sin \gamma \sin(\omega_i + \gamma)}{k(1, \zeta_i, \omega_i)}.$$

1.3 Specialisation $n = 1$

In the next chapter we will introduce the q -deformed Knizhnik–Zamolodchikov equation, which depends on q as well as an additional parameter s . At $s^4 q^3 = 1$ the solution of this equation is polynomial [52, 53], and in Section 2.1 we will use the interlacing conditions (1.2.14) to show that at $q^3 = s^4 = 1$ the ground state eigenvector of the $\text{TL}(n)$ loop model is a solution of the $q\text{KZ}$ equation.

The specialisation $q^3 = 1$ is equivalent to setting $n = 1$ or $\gamma = 2\pi/3$, and this also corresponds to anisotropy of the XXZ spin chain, $\Delta := \cos \gamma = -1/2$. In order to construct a special representation we will also set $\omega_0 = \omega_L = -2\gamma$ and $b = 1$. At this point we have several simplifications, the first of which is that k as given in (1.2.2) becomes

$$k(z, \zeta, \omega) = [z/q\zeta][z\zeta/q] =: k(z, \zeta). \quad (1.3.1)$$

As stated in Section 1.2.1, when $q = e^{2\pi i/3}$ we take the normalisation factor $\kappa(z)$ to be identically equal to 1. This choice for κ causes R to be invariant under negation of its arguments, so the crossing relation (1.2.11) can be written without the negative sign,

$$R(z, w) = R(qw, z).$$

When the above specialisations are taken, the constant in the expression for the

Hamiltonian (1.2.15) disappears, and the rest simplifies to

$$H = c_0(1 - e_0) + c_L(1 - e_L) + \sum_{j=1}^{L-1} (1 - e_j), \quad (1.3.2)$$

$$c_i = \frac{3}{1 + \zeta_i^2 + \zeta_i^{-2}}. \quad (1.3.3)$$

For convenience, we keep the notation q , and make use of the fact that $q^3 = 1$. We also see that the relations of the 2BTL algebra can now be expressed as

$$\begin{aligned} e_i^2 &= e_i, & \forall i, \\ e_i e_{i+1} e_i &= e_i, & 1 \leq i \leq L-1. \end{aligned}$$

This has a one-dimensional representation ρ defined by

$$\rho : e_i \mapsto 1, \quad i = 0, 1, \dots, L, \quad (1.3.4)$$

which is a quotient of the link pattern representation, as it maps every link pattern to the identity (this is easily seen by viewing each link pattern as a word in the e_i , as described in Section 1.1.2). We choose the values $\omega_0 = \omega_L = -2\gamma$ and $b = 1$ in order to construct this representation. We note that $\rho(H) = 0$, hence 0 is an eigenvalue of H in the link pattern representation. In fact, because the eigenvalues of e_i are 0 and 1, for c_0, c_L non-negative, 0 is the lowest eigenvalue of H and corresponds to the ground state of the $O(n = 1)$ loop model.

Moreover, we can use a version of the Perron–Frobenius theorem to show that the ground state is unique. Consider the matrix $\mathcal{H} = \mathbb{I}L - H$, which has non-negative entries for c_0, c_L non-negative, and which shares eigenvectors with H . The expansion of \mathcal{H}^m includes every possible word in the generators e_i of length m or shorter, so for large enough m , acting with \mathcal{H}^m on an arbitrary basis vector produces a vector with all entries non-zero. Thus \mathcal{H} is irreducible, and the Perron–Frobenius theorem states that the eigenvector corresponding to eigenvalue L (equal to the ground state of H) is unique. We will be interested in the ground state eigenvector $|\Psi\rangle$ as a function of the parameters c_0 and c_L ,

$$H |\Psi(c_0, c_L)\rangle = 0.$$

In the same way, the one-dimensional representation indicates that the ground state eigenvalue of the transfer matrix is equal to 1, so the eigenvalue equation is

$$T_L(w; z_1, \dots, z_L) |\Psi_L(z_1, \dots, z_L)\rangle = |\Psi_L(z_1, \dots, z_L)\rangle. \quad (1.3.5)$$

In the homogeneous limit $z_i \rightarrow 1$, the transfer matrix T_L becomes the probability matrix

of the stochastic raise and peel model [78, 81], for which the steady state eigenvector (1.3.5) is unique, again by the Perron–Frobenius theorem. The eigenvalue spectrum is continuous, so it can be argued that there exists an open set around $z_1 = \dots = z_L = 1$ for which the eigenvector remains unique. We will thus assume that the eigenvector remains unique for generic values of z_1, \dots, z_L .

The q KZ equation will allow us to obtain an explicit characterisation of $|\Psi_L\rangle$ for finite L . We will in particular be able to derive a closed form expression for the normalisation Z_L , which is the sum of all the components of $|\Psi_L\rangle$, as well as for a boundary-to-boundary correlation function defined in Chapter 3. In order to do so we need a recursion relation for $|\Psi_L\rangle$, which we will discuss in Section 2.2.2.

It is worth noting that in the limit $e_L \mapsto 1$ the boundary Baxterised element \check{K}_L maps to the identity, and this is the limit in which the two-boundary model maps to the one-boundary case. Similarly, in the limit $e_0, e_L \mapsto 1$, we obtain the trivial boundary case.

Chapter 2

The ground state of the TL(1) loop model

The aim of this chapter is to calculate the elements of the ground state eigenvector of the TL(1) loop model with two open boundaries, corresponding to the 2BTL algebra as discussed in the previous chapter. As will be shown, at $q^3 = 1$, the ground state eigenvalue equation (1.3.5) for the inhomogeneous transfer matrix of the TL(1) model is equivalent to a particular instance of the q -deformed Knizhnik–Zamolodchikov (q KZ) equation.

The q KZ equation is a set of finite difference equations, introduced by Frenkel and Reshetikhin [41] in the context of the representation theory of quantum affine algebras. This equation depends on a number of parameters, traditionally t, q (this is not the same as our q), and if applicable the boundary parameters a, b, c and d . It also has two extra parameters, k and r , called respectively the ‘rank’ and ‘level’ of the equation. It has been found that when these parameters satisfy

$$t^{k+1}q^{r-1}abcd = 1, \tag{2.0.1}$$

the q KZ equation has polynomial solutions [52, 53, 91] (see also [38, 39]).

Interesting recent developments [53] relate polynomial solutions of the q KZ equation associated with $U_q(sl_k)$ to the polynomial representation of the double affine Hecke algebra [51, 70, 88]. These solutions can be expressed in terms of Koornwinder or Macdonald polynomials with specialised parameters [1, 39].

As mentioned, the ground state of the TL(1) model is a solution of the q KZ equation with very specialised parameters. In particular, the parameters a, b, c and d are set to 1, the parameter t is equal to q^2 in our notation, and the parameter q is related to a new parameter s . We take both the rank k and the level r to be 2, and the restriction on the parameters (2.0.1) becomes $q^6s^4 = 1$. Since the ground state eigenvalue equation is

equivalent to the q KZ equation at $q^3 = 1$, this condition gives us the further restriction $s^4 = 1$. As we will see, the version of the q KZ equation used here has Laurent polynomials as solutions.

The ground states for $TL(1)$ models with a variety of boundary conditions have been studied in the past (see for example [23, 27, 66, 109]), and each one is related to a version of the q KZ equation. In the models with zero or one open boundaries, there exists a highest weight vector, which is an element of the link pattern basis. The component of the ground state eigenvector corresponding to this basis element plays a special role in the solution, because it is possible to specify this component exactly, and from there calculate every other element of the vector. In the case of periodic boundaries, due to the cyclic symmetry there is not a unique highest weight vector, but an equivalence class of them. Again the corresponding components of the eigenvector can be fixed, and the q KZ equation provides the means to then calculate the other components.

However, in the case of two open boundaries, there is no such highest weight vector. This fact causes the calculation of the ground state eigenvector in the case of two open boundaries to be more challenging, and until recently little progress had been made.

In this chapter we deal with this case, and generalise the results for reflecting [27] and mixed [109] boundary conditions. It is very hard to find a general expression for all the components of the two boundary eigenvector, but a certain subset of them can be found, and for the others certain restrictions can be made. We also calculate the normalisation of the eigenvector, which is the sum of all its components. This can be found explicitly despite the lack of an exact formula for all the elements of the eigenvector. The main results in this chapter were also discovered independently and through slightly different means by Cantini [10], in the same year.

For small system sizes $L \leq 2$, we solve the q KZ equation explicitly for all components, up to an overall factor that we take to be 1. For arbitrary size, we find an explicit expression for two special components of the eigenvector, using recursive properties of the $TL(1)$ transfer matrix. We then use the same recursive properties to find an expression for the overall ground state normalisation.

With the specialisation $s^4 = 1$, we find that the dependence on s factors out of the final result. However we keep the notation for s in the expressions for the components with a view to generalisation.

2.1 The q -deformed Knizhnik–Zamolodchikov equation

As stated above, the ground state eigenvector of the $TL(1)$ transfer matrix is a solution of the q KZ equation with a certain specialisation. This connection will provide the foundation for an explicit analysis of the ground state eigenvector for finite system size L . We will first describe the q KZ equation for open boundaries, and then prove the

equivalence with the transfer matrix eigenvalue equation (1.3.5).

We consider a linear combination $|\Xi\rangle$ of states $|\alpha\rangle$ labelled by link patterns:

$$|\Xi(z_1, \dots, z_L)\rangle = \sum_{\alpha} \xi_{\alpha}(z_1, \dots, z_L) |\alpha\rangle.$$

Here the sum runs over the set of link patterns of size L , and the coefficient functions ξ_{α} are Laurent polynomials in the variables z_1, \dots, z_L with coefficients that are functions of the boundary parameters ζ_0 and ζ_L as well as q and s .

Definition 2.1.1. The q -deformed Knizhnik–Zamolodchikov equation [41, 93] is a system of finite difference equations on the vector $|\Xi\rangle$. For open boundary conditions they can be written as [10, 14, 27, 109],*

$$\begin{aligned} \check{R}_i(z_i/z_{i+1}) |\Xi\rangle &= \pi_i |\Xi\rangle, & 1 \leq i \leq L-1, \\ \check{K}_0(1/z_1, \zeta_0) |\Xi\rangle &= \pi_0 |\Xi\rangle, \\ \check{K}_L(s z_L, s \zeta_L) |\Xi\rangle &= \pi_L |\Xi\rangle. \end{aligned} \tag{2.1.1}$$

This definition uses the Baxterised elements \check{R}_i , \check{K}_0 and \check{K}_L of the two-boundary Temperley–Lieb algebra, defined in (1.2.1). The operators $\check{R}_i(z_i/z_{i+1})$, $\check{K}_0(1/z_1, \zeta_0)$ and $\check{K}_L(s z_L, s \zeta_L)$ act on link patterns $|\alpha\rangle$, whereas the operators π_i ($i = 0, \dots, L$) act on the coefficient functions ξ only;

$$\begin{aligned} \pi_i \xi(\dots, z_i, z_{i+1}, \dots) &= \xi(\dots, z_{i+1}, z_i, \dots), & 1 \leq i \leq L-1, \\ \pi_0 \xi(z_1, \dots) &= \xi(1/z_1, \dots), \\ \pi_L \xi(\dots, z_L) &= \xi(\dots, 1/s^2 z_L). \end{aligned}$$

For later convenience, we note that the q KZ equations can be rewritten

$$e_i |\Xi\rangle = -a_i |\Xi\rangle, \quad 0 \leq i \leq L, \tag{2.1.2}$$

where e_i are the generators of the 2BTL from Definition 1.1.4, and

$$\begin{aligned} a_i &= (\pi_i + 1) \frac{[z_i/q z_{i+1}]}{[z_i/z_{i+1}]}, & 1 \leq i \leq L-1, \\ a_0 &= (\pi_0 + 1) \frac{k(1/z_1, \zeta_0)}{[q][z_1^2]}, \\ a_L &= -(\pi_L + 1) \frac{k(s z_L, s \zeta_L)}{[q][s^2 z_L^2]}, \end{aligned} \tag{2.1.3}$$

where k was defined in (1.3.1). The operators a_i ($i = 0, \dots, L$) satisfy the relations of

*We write the equations in a form used by Smirnov [93] for the type A case (see Appendix A of this thesis for more details on the type classifications).

the Hecke algebra of affine type C [23], as well as those of the Hecke algebra of type BC . The Hecke algebra is described in more detail in Appendix A.

For later convenience, we also define operators s_i , such that

$$\begin{aligned} s_i &= (q + q^{-1}) - a_i, & 1 \leq i \leq L-1, \\ s_k &= \frac{\sin \omega_k}{\sin(\omega_k + \gamma)} - a_k, & k = 0, L. \end{aligned} \quad (2.1.4)$$

With our chosen specifications for q , ω_0 , and ω_L (see Section 1.3), these definitions can be condensed to

$$s_i = -1 - a_i, \quad \forall i. \quad (2.1.5)$$

The s_i can be explicitly written as

$$\begin{aligned} s_i &= \frac{[qz_i/z_{i+1}]}{[z_i/z_{i+1}]} (1 - \pi_i), & 1 \leq i \leq L-1, \\ s_0 &= \frac{k(z_1, \zeta_0)}{[q][z_1^2]} (\pi_0 - 1), \\ s_L &= \frac{k(1/sz_L, s\zeta_L)}{[q][s^2z_L^2]} (1 - \pi_L). \end{aligned}$$

2.1.1 Equivalence with the transfer matrix eigenvalue equation

At the special values of q and s previously mentioned, the elements of the eigenvector are polynomials in z_1^\pm, \dots, z_n^\pm , and we use this fact to construct a definition for $|\Psi_L\rangle$ that is appropriately normalised.

Definition 2.1.2. With the specialisations listed in Section 1.3, the ground state eigenvector $|\Psi_L(z_1, \dots, z_L)\rangle$ of the transfer matrix of the TL(1) model of width L has eigenvalue 1, and is written in the link pattern basis as

$$|\Psi_L(z_1, \dots, z_L)\rangle = \sum_{\alpha \in \text{LP}_L} \psi_\alpha(z_1, \dots, z_L) |\alpha\rangle,$$

where the ψ_α are coprime polynomials.

The basis orthogonal to the downward link patterns $|\alpha\rangle$ is given by the row vectors $\langle\beta|$, with the usual inner product $\langle\cdot|\cdot\rangle$ defined by

$$\langle\beta|\alpha\rangle = \delta_{\alpha,\beta}.$$

Written in this basis, the left eigenvector $\langle\Psi|$ of the transfer matrix corresponding to eigenvalue 1 is a row vector with every element equal to 1. This is because the transfer matrix is a stochastic matrix, whose columns sum to 1.

Definition 2.1.3. The normalisation of the eigenvector is defined as

$$Z_L = \langle \Psi | \Psi \rangle,$$

or more explicitly,

$$\begin{aligned} Z_L(z_1, \dots, z_L) &= \sum_{\beta, \alpha} \psi_\alpha(z_1, \dots, z_L) \langle \beta | \alpha \rangle \\ &= \sum_{\alpha} \psi_\alpha(z_1, \dots, z_L). \end{aligned} \quad (2.1.6)$$

For the next theorem we will also need the following lemma.

Lemma 2.1.1. *In the almost homogeneous limit $z_1 \rightarrow q$, $z_2 = \dots = z_L = \zeta_0 = \zeta_L = 1$, the Hamiltonian (1.3.2) can be normalised and expressed as*

$$\tilde{H} = e_1 e_0 - 1.$$

The ground state eigenvector of this Hamiltonian, corresponding to eigenvalue 0, is non-negative.

Proof. At the point $z_2 = \dots = z_L = \zeta_0 = \zeta_L = 1$, we can calculate the Hamiltonian as was done in Section 1.2.3, obtaining

$$H = \left(\frac{3}{1 + z_1^2 + z_1^{-2}} - e_0 \right) + \sum_{i=1}^{L-1} (1 - e_i) + (1 - e_L) + \frac{[z_1]}{[q/z_1]} e_1 e_0 + \frac{[1/z_1]}{[qz_1]} e_0 e_1.$$

Normalising H by $\frac{[q/z_1]}{[z_1]}$, and taking the limit as $z_1 \rightarrow q$, this becomes

$$\begin{aligned} \tilde{H} &:= \lim_{z_1 \rightarrow q} \frac{[q/z_1]}{[z_1]} H = e_1 e_0 + \lim_{z_1 \rightarrow q} \frac{3 [q/z_1]}{(1 + z_1^2 + z_1^{-2}) [z_1]} \\ &= e_1 e_0 - 1. \end{aligned}$$

In matrix notation the non-diagonal entries of this Hamiltonian are non-negative, so according to the Perron–Frobenius theorem the ground state eigenvector must also be non-negative. \square

Theorem 2.1.2. *The specialised ground state eigenvector $|\Psi\rangle$ also satisfies the qKZ equation, Definition 2.1.1.*

The proof of this given in [22] was incomplete. We give here the full proof.

Proof. We first act on both sides of the eigenvector equation (1.3.5) with the Baxterised

element $\check{R}(z_i/z_{i+1})$, and using the interlacing condition in (1.2.14),

$$\begin{aligned}\check{R}_i(z_i/z_{i+1})|\Psi_L(z_i, z_{i+1})\rangle &= \check{R}_i(z_i/z_{i+1})T_L(w; z_i, z_{i+1})|\Psi_L(z_i, z_{i+1})\rangle \\ &= T_L(w; z_{i+1}, z_i)\check{R}_i(z_i/z_{i+1})|\Psi_L(z_i, z_{i+1})\rangle.\end{aligned}$$

Since the eigenvector in (1.3.5) is unique, this implies that

$$\check{R}_i(z_i/z_{i+1})|\Psi_L(z_i, z_{i+1})\rangle = \beta_i(z_1, \dots, z_L)|\Psi_L(z_{i+1}, z_i)\rangle, \quad (2.1.7)$$

where β_i is some rational function. By definition, none of the elements of $|\Psi\rangle$ have a denominator, so the denominator of $\beta_i(\dots, z_i, z_{i+1}, \dots)$ must be the same as the denominator of $\check{R}_i(z_i/z_{i+1})$, i.e., $[qz_i/z_{i+1}]$. We can rearrange the above, using the unitarity relation (1.2.7) of \check{R} , and swap the parameters $z_i \leftrightarrow z_{i+1}$ to get

$$\check{R}_i(z_i/z_{i+1})|\Psi_L(z_i, z_{i+1})\rangle = \beta_i(\dots, z_{i+1}, z_i, \dots)^{-1}|\Psi_L(z_{i+1}, z_i)\rangle, \quad (2.1.8)$$

which leads directly to

$$\beta_i(\dots, z_i, z_{i+1}, \dots) = \beta_i(\dots, z_{i+1}, z_i, \dots)^{-1}.$$

Since we already know the denominator of β_i , the above property gives us four choices for the numerator, namely

$$\beta_i(z_i, z_{i+1}) = \frac{\pm[qz_{i+1}/z_i]}{[qz_i/z_{i+1}]} \quad \text{or} \quad \pm 1. \quad (2.1.9)$$

In both cases the sign is fixed to +1 by setting $z_{i+1} = z_i$ in (2.1.8).

Assume β_i to be the first choice in (2.1.9). In the quotient $\rho : e_i \mapsto 1$ as defined in (1.3.4), every link pattern is projected onto the identity, and (2.1.7) becomes

$$Z_L(z_i, z_{i+1}) = \frac{[qz_{i+1}/z_i]}{[qz_i/z_{i+1}]} Z_L(z_{i+1}, z_i).$$

Since Z_L is a polynomial in the z 's, it must be of the form

$$Z_L = [qz_{i+1}/z_i] S_i(z_1, \dots, z_L),$$

where S_i is a polynomial symmetric in z_i, z_{i+1} . By considering the above argument for all other values of i we find that Z_L must be of the form

$$Z_L = \prod_{1 \leq i < j \leq L} [qz_j/z_i] S(z_1, \dots, z_L),$$

where S here is a polynomial symmetric in all its arguments. This means that the eigen-

vector normalisation vanishes when $z_i = qz_j$, for any $j > i$. However, the specialisation in Lemma 2.1.1 has $z_1 = qz_j$ for all $j > 1$, and it is shown that Z_L cannot be 0 at that point. Our assumption must therefore be false and we must take $\beta_i = 1$.

Thus, we have shown that

$$\check{R}_i(z_i/z_{i+1}) |\Psi_L(z_1, \dots, z_L)\rangle = \pi_i |\Psi_L(z_1, \dots, z_L)\rangle, \quad 1 \leq i \leq L-1.$$

Similarly, we can show that

$$\begin{aligned} \check{K}_0(1/z_1, \zeta_0) |\Psi_L(z_1, \dots, z_L)\rangle &= \pi_0 |\Psi_L(z_1, \dots, z_L)\rangle, \\ \check{K}_L(sz_L, s\zeta_L) |\Psi_L(z_1, \dots, z_L)\rangle &= \pi_L |\Psi_L(z_1, \dots, z_L)\rangle, \end{aligned}$$

where the proof of the last equation makes use of the fact that when $s^4 = 1$, $\check{K}_L(sa, sb) = \check{K}_L(a, b)$ and $R(s^2z, w) = R(z, w)$. \square

2.2 Recursions

We will use the q KZ equation (2.1.1) to calculate the elements of the ground state eigenvector of the TL(1) loop model for size L . However this equation does not contain enough information to fix the elements, so we will also use a recursion that is an inherent property of the loop model.

We begin by defining some maps between spaces of link patterns of different sizes. These maps will be very useful in constructing recursions satisfied by the transfer matrix and its eigenvector.

Definition 2.2.1. For $1 \leq i \leq L+1$, let φ_i be the map that takes a link pattern of size L , sends site j to $j+2$ for $j \geq i$, and then inserts a link from site i to $i+1$, thus creating a link pattern of size $L+2$. For example,

$$\varphi_6 : \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right| \mapsto \left| \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right|.$$

Let $\tilde{\varphi}_0$ be the map that takes a link pattern of size L , sends site j to site $j+1$ for all j , and inserts a ‘)’ on the first site.

Let $\tilde{\varphi}_{L+1}$ be the map that takes a link pattern of size L and inserts a ‘(’ after the last site.

We will also define similar maps on vectors in the link pattern space defined in Section 1.1.2. The action on the basis is a straightforward extension of the action on the link patterns, so we use the same notation for both maps.

Definition 2.2.2. Let $\varphi_i : \text{LP}_L \rightarrow \text{LP}_{L+2}$ be the map defined by

$$\varphi_i \sum_{\alpha} c_{\alpha} |\alpha\rangle = \sum_{\alpha} c_{\alpha} |\varphi_i \alpha\rangle.$$

Similarly let $\tilde{\varphi}_0 : \text{LP}_L \rightarrow \text{LP}_{L+1}$ and $\tilde{\varphi}_{L+1} : \text{LP}_L \rightarrow \text{LP}_{L+1}$ be defined by

$$\begin{aligned} \tilde{\varphi}_0 \sum_{\beta} c_{\beta} |\beta\rangle &= \sum_{\beta} c_{\beta} |\tilde{\varphi}_0 \beta\rangle, \\ \tilde{\varphi}_{L+1} \sum_{\beta} c_{\beta} |\beta\rangle &= \sum_{\beta} c_{\beta} |\tilde{\varphi}_{L+1} \beta\rangle. \end{aligned}$$

It is worth noting that the vector resulting from the action by φ_i is of length 2^{L+2} , but it has only 2^L non-zero entries, which are indexed by link patterns with a small link from i to $i+1$. Similar statements can be made for $\tilde{\varphi}_0$ and $\tilde{\varphi}_L$.

2.2.1 Transfer matrix recursion

Proposition 2.2.1. *The transfer matrix satisfies the following identity for general q :*

$$T_L(w; z_1, \dots, z_{i+1} = qz_i, \dots, z_L) \circ \varphi_i = \frac{[q/z_i w][q^2 z_i/w]}{[q^2 z_i w][qw/z_i]} \varphi_i \circ T_{L-2}(w; \dots, z_{i-1}, z_{i+2}, \dots).$$

The proof of this proposition is in Appendix B.1. A similar relation was proved in [27] and for the case of periodic boundary conditions in [29].

Proposition 2.2.2. *Likewise, at the boundaries, and for $q = e^{2\pi i/3}$, the transfer matrix satisfies*

$$T_L(w; z_1 = q\zeta_0, \dots, z_L; \zeta_0, \zeta_L) \circ \tilde{\varphi}_0 = \tilde{\varphi}_0 \circ T_{L-1}(w; z_2, \dots, z_L; q\zeta_0, \zeta_L), \quad (2.2.1)$$

$$T_L(w; z_1, \dots, z_L = \zeta_L/q; \zeta_0, \zeta_L) \circ \tilde{\varphi}_L = \tilde{\varphi}_L \circ T_{L-1}(w; z_1, \dots, z_{L-1}; \zeta_0, \zeta_L/q). \quad (2.2.2)$$

The first of these is proved in Appendix B.2.

2.2.2 Recursion of the eigenvector

In order to find a recursive definition for all components of $|\Psi_L\rangle$, we must refer to the recursive property of the transfer matrix described in Proposition 2.2.1. We will suppress the arguments z_1, \dots, z_L of T and $|\Psi_L\rangle$ except where detail is needed. The notation \hat{z}_j will mean that z_j is missing from the list z_1, \dots, z_L . When we specify q to be a third root of unity, the proportionality factor in Proposition 2.2.1 becomes 1, so

$$T_L(w; z_{i+1} = qz_i) \circ \varphi_i = \varphi_i \circ T_{L-2}(w; \hat{z}_i, \hat{z}_{i+1}). \quad (2.2.3)$$

Acting with both sides of this equation on the eigenvector $|\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle$,

$$T_L(w; z_{i+1} = qz_i) \left(\varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle \right) = \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle ,$$

which, by uniqueness of the eigenvector $|\Psi_L\rangle$, implies

$$|\Psi_L(z_{i+1} = qz_i)\rangle = p_i(z_i; z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L) \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle , \quad (2.2.4)$$

where p_i is a proportionality factor. This proportionality implies that any component corresponding to a link pattern without a small link connecting i and $i+1$ * vanishes when $z_{i+1} = qz_i$. In Section 2.4 this property of the eigenvector will be derived in another way. Relation (2.2.4) was already proved for subcases of the most general open boundary conditions in [27, 109], and for periodic boundary conditions in [29].

Likewise, from the boundary recursions (2.2.1) and (2.2.2) of the transfer matrix we deduce that

$$\begin{aligned} |\Psi_L(z_1 = q\zeta_0; \zeta_0)\rangle &= r_0(z_2, \dots, z_L; \zeta_0) \tilde{\varphi}_0 |\Psi_{L-1}(\hat{z}_1; q\zeta_0)\rangle , \\ |\Psi_L(z_L = \zeta_L/q; \zeta_L)\rangle &= r_L(z_1, \dots, z_{L-1}; \zeta_L) \tilde{\varphi}_L |\Psi_{L-1}(\hat{z}_L; \zeta_L/q)\rangle , \end{aligned}$$

where r_0 and r_L are proportionality factors analogous to p_i .

The above recursions for the eigenvector imply the following recursions for the components of the eigenvector.

Lemma 2.2.3.

$$\begin{aligned} \psi_{(\varphi_i \circ \alpha)}^L(z_{i+1} = qz_i) &= p_i(z_i; z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L) \psi_{\alpha}^{L-2}(\hat{z}_i, \hat{z}_{i+1}), \\ \psi_{(\tilde{\varphi}_0 \circ \beta)}^L(z_1 = q\zeta_0) &= r_0(z_2, \dots, z_L; \zeta_0) \psi_{\beta}^{L-1}(\hat{z}_1; q\zeta_0), \\ \psi_{(\tilde{\varphi}_L \circ \beta)}^L(z_L = q/\zeta_L) &= r_L(z_1, \dots, z_{L-1}; \zeta_L) \psi_{\beta}^{L-1}(\hat{z}_L; \zeta_L/q), \end{aligned}$$

where α is any link pattern of length $L - 2$ and β is any link pattern of length $L - 1$.

Clearly the normalisation Z given in (2.1.6) also satisfies these recursions. In Appendix C, we show that Z is symmetric in its arguments, and use this to prove that p_i is symmetric in all its variables except z_i , and that r_0 and r_L are symmetric in all the z_i . It is also shown that the function p_i takes the same form for each i . We henceforth drop the index i from p .

*We use the term ‘small’ or ‘little’ link to mean a link connecting neighbouring sites.

2.3 Solutions to the q KZ equation for small L

We now want to solve the q KZ equation to find the components of the ground state eigenvector. For this purpose we use the form of the equation given in (2.1.2). We will be looking for the lowest degree solution for which all the ψ_α are not identically zero. Any higher degree solution must be a scalar multiple of the lowest degree solution, because of the uniqueness of the $\text{TL}(1)$ ground state eigenvector.

For $L = 0$, the solution is trivial, but it will be needed for calculations of the recursions as described in Section 2.2.2. Obviously the eigenvector is of length one, i.e., a scalar. The definition of the eigenvector has the components being coprime, and so by analogy we take this scalar to be 1.

2.3.1 Example: $L = 1$

When $L = 1$, there are only two link patterns, denoted by $\langle \rangle$ and $\langle \cdot \rangle$. The ground state eigenvector of the $\text{TL}(1)$ loop model in the basis of link patterns is then

$$|\Psi\rangle = \psi_{\langle \rangle} |\langle \rangle\rangle + \psi_{\langle \cdot \rangle} |\langle \cdot \rangle\rangle.$$

The e_i act only on link patterns, leaving the polynomials unchanged. In particular, e_0 acts on both link patterns to produce $\langle \cdot \rangle$, and e_1 acts on both to produce $\langle \rangle$. Remembering that the a_i act only on polynomials, the q KZ equations can be rewritten as

$$\begin{aligned} (\psi_{\langle \cdot \rangle} + \psi_{\langle \rangle}) |\langle \rangle\rangle &= -a_0 \psi_{\langle \cdot \rangle} |\langle \cdot \rangle\rangle - a_0 \psi_{\langle \rangle} |\langle \rangle\rangle, \\ (\psi_{\langle \cdot \rangle} + \psi_{\langle \rangle}) |\langle \cdot \rangle\rangle &= -a_1 \psi_{\langle \cdot \rangle} |\langle \cdot \rangle\rangle - a_1 \psi_{\langle \rangle} |\langle \rangle\rangle. \end{aligned}$$

Recalling that $s_i = -1 - a_i$, $\forall i$, we can match up the coefficients of each link pattern to get a system of four equations,

$$a_0 \psi_{\langle \cdot \rangle} = 0, \quad s_0 \psi_{\langle \rangle} = \psi_{\langle \cdot \rangle}, \quad a_1 \psi_{\langle \cdot \rangle} = 0, \quad s_1 \psi_{\langle \rangle} = \psi_{\langle \cdot \rangle}.$$

These equations, with the definitions of a_i in (2.1.3), give us all the information we need to find the minimal degree solution. From the first equation, we know that

$$k(1/z_1, \zeta_0) \psi_{\langle \cdot \rangle}$$

is invariant under $z_1 \rightarrow 1/z_1$, and we can deduce from this invariance that $\psi_{\langle \cdot \rangle}$ must be of the form

$$\psi_{\langle \cdot \rangle} = k(z_1, \zeta_0) f_1(z_1),$$

where f_1 is a polynomial invariant under $z_1 \rightarrow 1/z_1$. Similarly, from the third equation we find that ψ_{\rangle} must be given by

$$\psi_{\rangle} = k(1/sz_1, s\zeta_L)\tilde{f}_1(z_1),$$

where \tilde{f}_1 is invariant under $sz_1 \rightarrow 1/sz_1$.

We find that the remaining two equations are satisfied if f_1 and \tilde{f}_1 are both constants with $\tilde{f}_1 = s^2 f_1$, and this gives us the lowest degree solution. We want the components to be coprime, so we choose f_1 to be -1 ,^{*} and the solution for $L = 1$ is

$$\begin{aligned}\psi_{\langle} &= -k(z_1, \zeta_0), \\ \psi_{\rangle} &= -s^2 k(1/sz_1, s\zeta_L).\end{aligned}\tag{2.3.1}$$

Recursion to $L = 0$

According to the recursions listed in Lemma 2.2.3, specialising the above solutions at $z_1 = \zeta_L/q$ and $z_1 = q\zeta_0$ respectively will produce the boundary proportionality factors r_L and r_0 for $L = 1$ (recall that the solution for $L = 0$ is simply 1). This gives

$$\begin{aligned}r_L(\zeta_L) &= -s^2 k(1/s\zeta_L, s\zeta_0), \\ r_0(\zeta_0) &= -k(\zeta_0, \zeta_L).\end{aligned}\tag{2.3.2}$$

2.3.2 Example: $L = 2$

For $L = 2$, the eigenvector is of length four, and the link pattern basis consists of the elements $\{ |(\langle), |(\rangle), |)\langle), |)\rangle \}$. As before, in the q KZ equations the coefficients of each basis element can be collected together to form a system of equations. As an example, we take $i = 0$ in (2.1.2), and obtain

$$\begin{aligned}-\sum_{\alpha} a_0 \psi_{\alpha} |\alpha\rangle &= \sum_{\alpha} \psi_{\alpha} e_0 |\alpha\rangle \\ &= (\psi_{\langle\langle} + \psi_{\rangle\langle}) |)\langle\rangle + (\psi_{\langle\rangle} + \psi_{\rangle\langle}) |)\rangle.\end{aligned}$$

These equations can be written as

$$\begin{aligned}0 &= a_0 \psi_{\langle\rangle} = a_0 \psi_{\rangle\langle}, \\ \psi_{\langle\rangle} &= s_0 \psi_{\rangle\langle}, \quad \psi_{\rangle\langle} = s_0 \psi_{\langle\rangle}.\end{aligned}\tag{2.3.3}$$

^{*}We choose -1 instead of 1 with the benefit of hindsight — at this stage it makes no difference to the solution, but this choice results in a neater expression for the final solution for general L .

The rest of the system for $L = 2$ is

$$\begin{aligned} 0 &= a_2 \psi_{()} = a_2 \psi_{)}), \\ \psi_{()} &= s_2 \psi_{((}, \quad \psi_{)}) = s_2 \psi_{(} \quad , \end{aligned} \tag{2.3.4}$$

$$\begin{aligned} 0 &= a_1 \psi_{)}) = a_1 \psi_{(} = a_1 \psi_{((} \quad , \\ s_1 \psi_{()} &= \psi_{)}) + \psi_{((} + \psi_{(} \quad . \end{aligned} \tag{2.3.5}$$

As in the $L = 1$ case, it is an easy consequence of the equation $a_0 \psi_{()} = 0$ that if $\psi_{()} \neq 0$, it should contain a factor $k(z_1, \zeta_0)$. Such vanishing conditions hold for all the components.

Proposition 2.3.1.

- i. $\psi_{()}$ and $\psi_{((}$ vanish or contain a factor $k(z_1, \zeta_0)$, the remainder being invariant under $z_1 \leftrightarrow 1/z_1$.
- ii. $\psi_{()}$ and $\psi_{)})$ vanish or contain a factor $k(1/sz_2, s\zeta_L)$, the remainder being invariant under $sz_2 \leftrightarrow 1/sz_2$.
- iii. $\psi_{)})$, $\psi_{(}$ and $\psi_{((}$ vanish or contain a factor $[qz_1/z_2]$, the remainder being a symmetric function in z_1 and z_2 .

Solution

With the known factors and symmetries from Proposition 2.3.1 above, we thus look for a solution of the form

$$\begin{aligned} \psi_{((} &= \prod_{i=1}^2 k(z_i, \zeta_0) \times [qz_1/z_2][q/z_1z_2] \times f_2(z_1, z_2), \\ \psi_{)}) &= \prod_{i=1}^2 k(1/sz_i, s\zeta_L) \times [qz_1/z_2][qs^2z_1z_2] \times \tilde{f}_2(z_1, z_2), \end{aligned}$$

where $f_2(z_1, z_2)$ is a symmetric function invariant under $z_i \leftrightarrow 1/z_i$, and $\tilde{f}_2(z_1, z_2)$ is symmetric and invariant under $sz_i \leftrightarrow 1/sz_i$. Note that with k as defined in (1.3.1) we could write this as

$$\begin{aligned} \psi_{((} &= \prod_{i=1}^2 k(z_i, \zeta_0) \times k(z_2, z_1) \times f_2(z_1, z_2), \\ \psi_{)}) &= \prod_{i=1}^2 k(1/sz_i, s\zeta_L) \times k(1/sz_1, sz_2) \times \tilde{f}_2(z_1, z_2). \end{aligned}$$

The other two components may be determined from

$$\psi_{()} = s_0 \psi_{)}, \quad \psi_{()} = s_1 \psi_{()} - \psi_{()} - \psi_{()} \quad (2.3.6)$$

We pause here to define an important polynomial, which will appear throughout the rest of this thesis. The symplectic character χ_λ of degree $\lambda = (\lambda_1, \dots, \lambda_L)$ is defined by

$$\chi_\lambda(z_1, \dots, z_L) = \frac{\det_L \left[z_i^{\lambda_j + L - j + 1} - z_i^{-\lambda_j - L + j - 1} \right]}{\det_L \left[z_i^{L - j + 1} - z_i^{-L + j - 1} \right]}. \quad (2.3.7)$$

For convenience, we use the notation

$$\tau_\lambda(z_1, \dots, z_L) = \chi_\lambda(z_1^2, \dots, z_L^2). \quad (2.3.8)$$

The symplectic character is symmetric, and for the degree $\lambda^{(L)}$ with $\lambda_j^{(L)} = \left\lfloor \frac{L-j}{2} \right\rfloor$ it satisfies the recursion

$$\tau_{\lambda^{(L)}}(z_1, \dots, z_L)|_{z_j=qz_i} = (-1)^L \prod_{k \neq i, j} k(z_i, z_k) \tau_{\lambda^{(L-2)}}(\hat{z}_i, \hat{z}_j). \quad (2.3.9)$$

The classical character χ_λ , or equivalently the Schur polynomial of the type C root system, with degree $\lambda = \lambda^{(L)}$ appears repeatedly in related studies on loop models [27, 109] and symmetry classes of alternating sign matrices [71].

When $s^4 = 1$, $q = e^{2\pi i/3}$ we find that the solution to equations (2.3.3)–(2.3.5) can be given explicitly by

$$\begin{aligned} f_2(z_1, z_2) &= -\tau_{(1,0,0)}(z_1, z_2, \zeta_L), \\ \tilde{f}_2(z_1, z_2) &= -\tau_{(1,0,0)}(s\zeta_0, sz_1, sz_2). \end{aligned}$$

Recursion to $L = 0$

From Condition (iii) of Proposition 2.3.1, when we set $z_2 = qz_1$, all components vanish except for $\psi_{()}$. From (2.3.6),

$$\begin{aligned} \psi_{()}|_{z_2=qz_1} &= (-1 - a_0)\psi_{()}|_{z_2=qz_1} \\ &= \pi_0 \frac{k(1/z_1, \zeta_0)}{[q][z_1^2]} k(1/sz_1, s\zeta_L) k(1/sz_2, s\zeta_L) k(1/sz_1, sz_2) \\ &\quad \times \tau_{(1,0,0)}(sz_1, sz_2, s\zeta_0)|_{z_2=qz_1} \\ &= k(z_1, \zeta_0)^2 k(z_1, \zeta_L)^2, \end{aligned}$$

where we have used the properties of k

$$k(s^2a, b) = s^2k(sa, sb) = k(a, 1/b) = k(1/qa, b) = k(a, b). \quad (2.3.10)$$

Since the solution for $L = 0$ is simply 1, we can easily see that the proportionality factor p in Lemma 2.2.3 for $L = 2$ is

$$p(z_1) = k(z_1, \zeta_0)^2 k(z_1, \zeta_L)^2. \quad (2.3.11)$$

Recursion to $L = 1$

Similarly, when we set $z_1 = q\zeta_0$, the components $\psi_{()}$ and $\psi_{((}$ vanish, and $\psi_{))}$ becomes

$$\psi_{))}|_{z_1=q\zeta_0} = s^2 k(1/sz_2, s\zeta_L) k(\zeta_0, \zeta_L) k(\zeta_0, z_2)^2.$$

Using our solution for $\psi_{))}(z_2)$ from (2.3.1), we deduce that

$$r_0(z_2; \zeta_0) = -k(\zeta_0, \zeta_L) k(\zeta_0, z_2)^2. \quad (2.3.12)$$

In the same way, setting $z_2 = \zeta_L/q$ in $\psi_{((}$ gives us

$$r_L(z_1; \zeta_L) = -s^2 k(1/s\zeta_L, s\zeta_0) k(1/s\zeta_L, sz_1)^2. \quad (2.3.13)$$

2.3.3 Example: $L = 3$

It is computationally very intensive to compute explicitly the full solution for $L = 3$. However, we can compute a solution for the subset of equations where $\psi_{)((}$ and $\psi_{))((}$ are not individually determined, but only their sum is (see Appendix D for more details). We find

$$\begin{aligned} \psi_{(((} &= \prod_{0 \leq i < j \leq 3} k(z_j, z_i) \tau_{(1,1,0,0)}(z_1, z_2, z_3, \zeta_L) g_3(z_1, z_2, z_3), \\ \psi_{)))} &= s^2 \prod_{1 \leq i < j \leq 4} k(1/sz_i, sz_j) \tau_{(1,1,0,0)}(s\zeta_0, sz_1, sz_2, sz_3) g_3(sz_1, sz_2, sz_3), \end{aligned}$$

with τ as before, and where g_3 is symmetric and invariant under $z_i \leftrightarrow 1/z_i$. We have also introduced the notation $z_0 = \zeta_0$ and $z_{L+1} = \zeta_L$. Imposing the boundary recursions in Lemma 2.2.3 requires that the lowest degree solution is

$$g_3(z_1, z_2, z_3) = \tau_{(1,0,0)}(z_1, z_2, z_3).$$

Recursion to $L = 2$

The above calculation also gives us

$$\begin{aligned} r_0(z_2, z_3; \zeta_0) &= -k(\zeta_0, \zeta_L)k(\zeta_0, z_2)^2k(\zeta_0, z_3)^2, \\ r_L(z_1, z_2; \zeta_L) &= -s^2k(1/s\zeta_L, s\zeta_0)k(1/s\zeta_L, sz_1)^2k(1/s\zeta_L, sz_2)^2. \end{aligned} \quad (2.3.14)$$

Recursion to $L = 1$

Computing $\psi_{()} = s_3\psi_{((}$, and setting $z_3 = qz_2$, we find the recursion between size $L = 3$ and size $L = 1$:

$$\begin{aligned} p(z_2; z_1) &= k(z_2, \zeta_0)^2k(z_2, \zeta_L)^2k(z_2, z_1)^4, \\ \Rightarrow p(z_i; z_j) &= k(z_i, \zeta_0)^2k(z_i, \zeta_L)^2k(z_i, z_j)^4, \quad j \neq i, i+1. \end{aligned} \quad (2.3.15)$$

2.4 Solution for general L

As in the case of $L = 2$ in Section 2.3.2, for general L we may derive factors for certain components. For each i from 1 to $L-1$, every link pattern in the LHS of the q KZ equation (2.1.2) will have a small link from i to $i+1$ once e_i has acted. The q KZ equation then says that $a_i\psi_\alpha = 0$ if α does not have a small link from i to $i+1$. This leads to the following conditions on ψ_α .

Proposition 2.4.1.

- i. If α does not have a small link from the left boundary to 1, ψ_α vanishes or contains a factor $k(z_1, \zeta_0)$, the remainder being invariant under $z_1 \leftrightarrow 1/z_1$.
- ii. If α does not have a small link from L to the right boundary, ψ_α vanishes or contains a factor $k(1/sz_L, s\zeta_L)$, the remainder being invariant under $sz_L \leftrightarrow 1/sz_L$.
- iii. If α does not have a small link from i to $i+1$, ψ_α vanishes or contains a factor $[qz_i/z_{i+1}]$, the remainder being a symmetric function in z_i and z_{i+1} .

Using the above conditions, for general L the component $\psi_{(...}$ is given by

$$\psi_{(...} = \prod_{0 \leq i < j \leq L} k(z_j, z_i) f_L(z_1, \dots, z_L), \quad (2.4.1)$$

where f_L is symmetric and invariant under $z_i \rightarrow 1/z_i$. The majority of the factors in this expression are imposed by the symmetry conditions.

Likewise, the component $\psi_{)...) is expressed as$

$$\psi_{)...) = \prod_{1 \leq i < j \leq L+1} k(1/sz_i, sz_j) \tilde{f}_L(sz_1, \dots, sz_L), \quad (2.4.2)$$

where \tilde{f}_L is symmetric and invariant under $sz_i \rightarrow 1/sz_i$. Other components may be derived from the extremal components by acting with products of Baxterised versions of the operators s_i , as described in Appendix A [23]. However, in the case under consideration it is not possible to derive every component of $|\Psi\rangle$ in this way. In Appendix D we explain the reasons for this in detail for the case $L = 3$.

To get more information about the polynomials f_L and \tilde{f}_L , as well as about the other components, we use the recursive properties of the eigenvector, as we have previously for small L . As before, the recursions come from Lemma 2.2.3, which relates components of the eigenvector for L to components of the eigenvector for $L - 1$ and $L - 2$.

2.4.1 Recursions

We have found the proportionality factors for small system sizes, given in (2.3.2), (2.3.11), (2.3.12), (2.3.13), (2.3.14) and (2.3.15). We make the assumption that these factors continue their pattern for larger system sizes, as stated in the following conjecture.

Conjecture 2.4.1. For general L the proportionality factors are

$$\begin{aligned} p(z_i; z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L) &= k(z_i, \zeta_0)^2 k(z_i, \zeta_L)^2 \prod_{j \neq i, i+1} k(z_i, z_j)^4, \\ r_0(z_2, \dots, z_L; \zeta_0) &= -k(\zeta_0, \zeta_L) \prod_{i=2}^L k(\zeta_0, z_i)^2, \\ r_L(z_1, \dots, z_{L-1}; \zeta_L) &= -s^2 k(1/s\zeta_L, s\zeta_0) \prod_{i=1}^{L-1} k(1/s\zeta_L, sz_i)^2. \end{aligned}$$

Using these factors, we can come up with associated recursions for the symmetric functions f_L and \tilde{f}_L . For example, we can calculate the size L eigenvector component $\psi_{(\dots)_0}^L$ using the q KZ equation and then, according to Lemma 2.2.3, setting $z_L = qz_{L-1}$ will give us $p(z_{L-1}; z_1, \dots, z_{L-2})$ multiplied by the size $L - 2$ component $\psi_{(\dots)_0}^{L-2}$. Recalling the definition (2.1.5) of s_i , we have from the q KZ equation that

$$\psi_{(\dots)_0}^L = s_L \psi_{(\dots)_0}^L,$$

and since $\psi_{(\dots)}^L|_{z_L=qz_{L-1}} = 0$, it follows that

$$\begin{aligned} \psi_{(\dots)}^L|_{z_L=qz_{L-1}} &= \left(\pi_L \frac{k(sz_L, s\zeta_L)}{[q][s^2z_L^2]} \prod_{0 \leq i < j \leq L} k(z_j, z_i) f_L(z_1, \dots, z_L) \right) \Big|_{z_L=qz_{L-1}} \\ &= k(z_{L-1}, \zeta_L) \prod_{i=0}^{L-2} k(z_{L-1}, z_i)^2 \prod_{0 \leq i < j \leq L-2} k(z_j, z_i) f_L(z_1, \dots, z_{L-1}, s^2qz_{L-1}) \\ &= k(z_{L-1}, \zeta_0)^2 k(z_{L-1}, \zeta_L) \prod_{i=1}^{L-2} k(z_{L-1}, z_i)^2 \frac{f_L(z_1, \dots, z_{L-1}, s^2qz_{L-1})}{f_{L-2}(z_1, \dots, z_{L-2})} \psi_{(\dots)}^{L-2}. \end{aligned}$$

Here we have used the properties of k given in (2.3.10). From above, the proportionality factor in this relation is given by $p(z_{L-1}; z_1, \dots, z_{L-2})$, so we arrive at a recursion for f_L ,

$$f_L(z_1, \dots, z_{L-1}, s^2qz_{L-1}) = k(z_{L-1}, \zeta_L) \prod_{j=1}^{L-2} k(z_{L-1}, z_j)^2 f_{L-2}(z_1, \dots, z_{L-2}).$$

Similarly, we can use the recursion from $\psi_{(\dots)}^L$ to $\psi_{(\dots)}^{L-2}$ to find a recursion for \tilde{f}_L . Due to the symmetry properties of both these functions, the recursions can be generalised to arbitrary i ,

$$\begin{aligned} f_L(z_1, \dots, z_i, s^2qz_i, \dots, z_L) &= k(z_i, \zeta_L) \prod_{j \neq i, i+1} k(z_i, z_j)^2 f_{L-2}(z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L), \\ \tilde{f}_L(z_1, \dots, z_i, s^2qz_i, \dots, z_L) &= k(z_i, \zeta_0) \prod_{j \neq i, i+1} k(z_i, z_j)^2 \tilde{f}_{L-2}(z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L). \end{aligned} \tag{2.4.3}$$

The boundary recursions from Lemma 2.2.3 can be immediately applied to the extremal components (2.4.1) and (2.4.2), and we find that f_L and \tilde{f}_L in addition satisfy

$$\begin{aligned} f_L(z_1, \dots, z_{L-1}, \zeta_L/q; \zeta_0, \zeta_L) &= - \prod_{j=1}^{L-1} k(1/\zeta_L, z_j) f_{L-1}(z_1, \dots, z_{L-1}; \zeta_0, \zeta_L/q), \\ \tilde{f}_L(q\zeta_0, z_2, \dots, z_L; \zeta_0, \zeta_L) &= -s^2 \prod_{j=2}^L k(\zeta_0, z_j) \tilde{f}_{L-1}(z_2, \dots, z_L; q\zeta_0, \zeta_L), \end{aligned} \tag{2.4.4}$$

where we have explicitly indicated the dependencies on ζ_0 and ζ_L .

2.4.2 Degree

Polynomial solutions of the q KZ can be labelled by their top degree μ , where μ is a partition, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_L \geq 0$. These solutions are of the form

$$\sum_{\nu \in \widetilde{W} \cdot \mu} c_\nu z^{2\nu},$$

where the notation z^ν stands for the product $z_1^{\nu_1} \dots z_L^{\nu_L}$, the coefficients c_ν are polynomials in ζ_0 and ζ_L , and $\widetilde{W} \cdot \mu$ denotes the orbit of μ under the action of the group \widetilde{W} , defined as follows. The generators of \widetilde{W} act on an L -tuple μ by the transformations

$$\begin{aligned} t_i : (\dots, \mu_i, \mu_{i+1}, \dots) &\mapsto (\dots, \mu_{i+1}, \mu_i, \dots), & 1 \leq i \leq n-1, \\ t_0 : (\mu_1, \mu_2, \dots) &\mapsto (1 - \mu_1, \mu_2, \dots), & \text{if } \mu_1 > 0, \\ t_L : (\mu_1, \mu_2, \dots) &\mapsto (\dots, \mu_{L-1}, -\mu_L), \end{aligned}$$

where t_0 acts as the identity if $\mu_1 < 0$. In effect, $W \cdot \mu$ is the set of all L -tuples that can be obtained from μ by any combination of the above actions.

We can use the recursions (2.4.3) and (2.4.4) to find the minimal degree of f_L and \tilde{f}_L for arbitrary size. The argument here is for f_L , but it is easily seen that it holds for \tilde{f}_L as well, and therefore that they have the same degree. In (2.4.3), consider $i = 1$ and denote the top degree of f_L by $\nu^{(L)} = (\nu_1^{(L)}, \dots, \nu_L^{(L)}, 0, 0, \dots)$. The (top) degree of $k(z_1, z_j)$ is $(1, 0)$ in the variables z_1^2 and z_j^2 , so the degree in z_1^2 on the right hand side of (2.4.3) is $2L - 3$. Since the degree in z_1^2 on the LHS must be the same, the added degrees in z_1^2 and z_2^2 of $f_L(z_1, \dots, z_L)$ must be greater than or equal to $2L - 3$.

In addition, by comparing degrees of any z_j^2 in (2.4.4), it immediately follows that $\nu_j^{(L)}$ is at least equal to $\nu_j^{(L-1)} + 1$. We thus find that the following inequalities must hold,

$$\begin{aligned} \nu_1^{(L)} + \nu_2^{(L)} &\geq 2L - 3, \\ \nu_j^{(L)} &\geq \nu_j^{(L-1)} + 1. \end{aligned}$$

For a minimal degree solution these inequalities become equalities, and using the solutions we explicitly constructed for the small system sizes $L = 1, 2, 3$ in Section 2.3, we find that

$$\nu_j^{(L)} = L - j \quad (j = 1, \dots, L).$$

We will write $\nu^{(L)} = \lambda^{(L)} + \lambda^{(L+1)}$, where $\lambda^{(L)}$ is the partition of $|\lambda^{(L)}| = \lceil \frac{L}{2} (\frac{L}{2} - 1) \rceil$ with

$$\lambda_j^{(L)} = \left\lfloor \frac{L-j}{2} \right\rfloor \quad j = 1, \dots, L, \quad (2.4.5)$$

i.e.,

$$\lambda^{(2n)} = (n-1, n-1, \dots, 1, 1, 0, 0), \quad \lambda^{(2n+1)} = (n, n-1, n-1, \dots, 1, 1, 0, 0).$$

From the degree of $k(z_j, z_i)$ it immediately follows that the product of factors in the expressions for the extremal components, $\psi_{(\dots(}$ and $\psi)_{\dots})$, amount to a degree of $\lambda^{(L+1)} + \lambda^{(L+2)}$. Solutions of the q KZ equation of minimal degree, which are relevant for the TL(1) loop model with open boundaries, therefore have degree $\mu^{(L)}$, with

$$\mu^{(L)} = \lambda^{(L)} + 2\lambda^{(L+1)} + \lambda^{(L+2)},$$

so that

$$\mu_j^{(L)} = 2L + 1 - 2j. \quad (2.4.6)$$

The total degree of these solutions is equal to $|\mu^{(L)}| = L^2$ and the degree in each variable z_i^2 is equal to $\mu_1 = 2L - 1$.

2.4.3 Eigenvector

By using recursion and degree properties of the general solution, we can find expressions for \tilde{f}_L and f_L . We emphasise again that we have taken $s^4 = 1$ and $q = e^{2\pi i/3}$.

For $L = 2$ and $L = 3$, the solution contains a symmetric function that involves the symplectic character defined in (2.3.7). The solution for general L can also be expressed in terms of this symplectic character. It turns out that the following two functions satisfy the necessary recursions (2.4.3) and (2.4.4), agree with the small size solutions, and have the correct degree $\nu^{(L)}$,

$$\begin{aligned} f_L(z_1, \dots, z_L) &= (-1)^{\frac{L(L+1)}{2}} \tau_{\lambda^{(L+1)}}(z_1, \dots, z_L, \zeta_L) \tau_{\lambda^{(L)}}(z_1, \dots, z_L), \\ \tilde{f}_L(sz_1, \dots, sz_L) &= (-1)^{\frac{L(L+1)}{2}} (s^2)^L \tau_{\lambda^{(L+1)}}(s\zeta_0, sz_1, \dots, sz_L) \tau_{\lambda^{(L)}}(sz_1, \dots, sz_L). \end{aligned} \quad (2.4.7)$$

It is worthwhile noting that (2.4.3) and (2.4.4) are satisfied because of the recursion for the symplectic character in (2.3.9), and the specification $s^4 = 1$.

2.4.4 Eigenvector normalisation

We have derived in Appendix C, see (C.1) and (C.2), the recursions for the normalisation Z_L as defined in (2.1.6). Using the recursion (2.3.9) for the symplectic character defined

in (2.3.7), we note that

$$\begin{aligned}
& \tau_{L+2}(\zeta_0, z_1, \dots, z_L, \zeta_L) \tau_{L+1}(\zeta_0, z_1, \dots, z_L) \tau_{L+1}(z_1, \dots, z_L, \zeta_L) \tau_L(z_1, \dots, z_L) |_{z_{i+1}=qz_i} \\
&= k(z_i, \zeta_0)^2 k(z_i, \zeta_L)^2 \prod_{j \neq i, i+1} k(z_i, z_j)^4 \tau_L(\zeta_0, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, \zeta_L) \\
&\quad \times \tau_{L-1}(\zeta_0, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) \tau_{L-1}(\dots, \hat{z}_i, \hat{z}_{i+1}, \dots, \zeta_L) \tau_{L-2}(\dots, \hat{z}_i, \hat{z}_{i+1}, \dots).
\end{aligned}$$

This product of four symplectic characters therefore satisfies the same recursions as Z_L (the boundary recursions are also easy to show). In addition, for $L = 1$ and $L = 2$ this product is equal to Z_1 and Z_2 , respectively. Since the recursions (C.1) and (C.2) specify enough points to uniquely determine Z_L of degree $\mu^{(L)}$, see (2.4.6), up to a factor independent of L , we conclude that

$$\begin{aligned}
Z_L(z_1, \dots, z_L) &= \tau_{L+2}(\zeta_0, z_1, \dots, z_L, \zeta_L) \tau_{L+1}(\zeta_0, z_1, \dots, z_L) \\
&\quad \times \tau_{L+1}(z_1, \dots, z_L, \zeta_L) \tau_L(z_1, \dots, z_L).
\end{aligned} \tag{2.4.8}$$

In particular, the normalisation of the ground state of the Hamiltonian (1.3.2) is obtained by setting $z_i = 1$, and is given by

$$Z_L = \tilde{Z}_2(c_0, c_L) \tilde{Z}_1(c_0) \tilde{Z}_1(c_L) \tilde{Z}_0, \tag{2.4.9}$$

where

$$\begin{aligned}
\tilde{Z}_0 &= \tau_L(1, \dots, 1), \\
\tilde{Z}_1(c_i) &= \tau_{L+1}(\zeta_i, 1, \dots, 1), \\
\tilde{Z}_2(c_0, c_L) &= \tau_{L+2}(\zeta_0, 1, \dots, 1, \zeta_L),
\end{aligned}$$

and c_i is defined as

$$c_i = \frac{3}{1 + \zeta_i^2 + \zeta_i^{-2}},$$

as in (1.3.3).

Chapter 3

The boundary-to-boundary correlation function

In an attempt to rigorously prove conformal invariance of the scaling limit of critical two-dimensional lattice models, an interesting new class of parafermionic observables for Potts and loop models was recently introduced, see for example [44, 86, 94, 96]. These observables are expressed in terms of the loop representation of the Fortuin–Kasteleyn cluster expansion of the Potts model [4, 5], and can be shown to be discretely holomorphic for certain parameter values. The operators corresponding to these observables carry a spin conjugate to the winding angles of the loops. In this chapter we will compute an exact closed form expression for the expectation of one such type of observable, with spin one, for the two-boundary TL(1) loop model as described in Chapter 1. The remarkable aspect of this result is the feasibility of an exact calculation of an expectation for a finite size model that is not free fermionic.

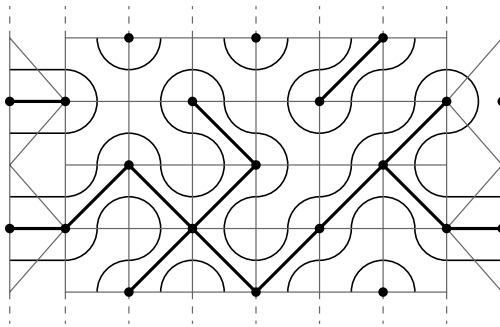


Figure 3.1: A percolating cluster with its associated hulls drawn as loops.

We consider the TL(1) loop model as an alternative description of the two-dimensional critical bond percolation model [77]. In this setting the percolation model is defined on a finite width, vertically infinite square lattice, as in Figure 3.1, where bonds are placed diagonally across the faces with a certain probability. Percolation occurs when a cluster

of bonds stretches from the left boundary to the right. As previously mentioned, the loops in the TL(1) model correspond to hulls drawn around the bond clusters, so a loop stretching from the left to the right implies a percolating cluster.

The expectation value calculated in this chapter can be interpreted as the density of percolation clusters passing in between two chosen vertices of the lattice, x_1 and x_2 [12, 34, 66, 95]. The analogous calculation for the density of clusters closed around a cylinder has also been performed [69]. We will sometimes refer to this density as the boundary-to-boundary correlation function for the two-boundary TL(1) model. Our calculation is valid for odd and even L , however for odd L the boundary conditions of the TL(1) model cause the interpretation in terms of the percolation model to break down.

This expectation value also corresponds to the spin current in a generalised Chalker–Coddington network model [13, 43, 64] for the quantum spin Hall effect. The Chalker–Coddington network model is based on the semi-classical picture of electrons in two dimensions moving under the influence of a strong perpendicular magnetic field in a long-ranged disorder potential. The network consists of a square lattice whose edges are unidirectional channels and whose vertices are scattering centres. A potential correlation length much larger than the magnetic length leads to the formation of clusters where the wave function amplitude changes only slightly. These clusters correspond to the clusters of bonds described above. The classical limit of the generalised Chalker–Coddington model [64] on the square lattice is therefore described by the solvable TL(1) lattice model. In a further generalisation, Gruzberg et al. [43] used a pseudo spin description of the particle and hole states. When the $SU(2)$ spin symmetry is observed, the full quantum mechanical spin current corresponds to the observable we study here.

In this chapter we will exactly calculate the correlation function for finite system size L . We will first give an explicit description of this quantity, before defining it as an expectation value in Section 3.1.1. We then discuss the symmetries and recursion relations satisfied by the function, and we use these properties along with a degree argument to prove an explicit form for the correlation function, given in Section 3.5. Our proof is complete except for two conjectures, both of which we have observed to be true for small L , and which we expect to hold in general.

3.1 Definition of the correlation function

Given a configuration C of bonds on the lattice, and any two marked vertices (at positions x_1 and x_2), we consider all the loops (hulls) that pass between these two vertices. We give a weight of 0 to those that close on themselves or have both ends connected to the same boundary, and a non-zero weight to loops that connect the left boundary to the right. The weight given is either 1 or -1 depending on the winding of the loop, so that a percolating cluster with a hull that crosses back and forth between the marked vertices is

only counted once. The sum of the loop weights gives the number of percolating clusters passing in between x_1 and x_2 in the configuration, multiplied by the sign relating to the winding. The average of this value over all configurations, weighted by the probability of obtaining each configuration, then gives the average density of percolating clusters between the marked points, which we refer to as the correlation function $F^{\{x_1, x_2\}}$.

This correlation function is given explicitly by

$$F^{\{x_1, x_2\}} = \sum_{C \in \Gamma} P(C) N_C^{\{x_1, x_2\}} \text{sign}_C^{\{x_1, x_2\}}. \quad (3.1.1)$$

Here Γ is the set of configurations, and the probability $P(C)$ of a configuration C is the product of weights of all the individual face configurations as defined by the R and K -matrices in Section 1.2.1.* $N_C^{\{x_1, x_2\}}$ is the number of paths passing in between the points x_1 and x_2 and running from the left to the right boundary, and $\text{sign}_C^{\{x_1, x_2\}}$ is $+1$ if x_1 lies in the region above the paths, and -1 if it lies below. The sign of the path in Figure 3.2 thus is -1 .

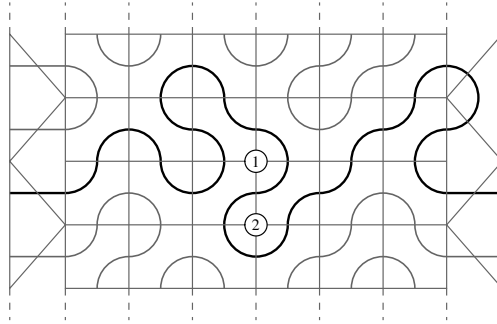


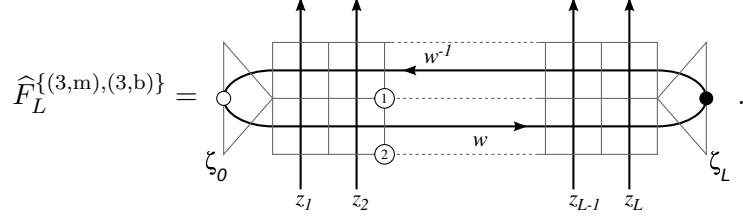
Figure 3.2: Path between markers 1 and 2 (at positions x_1 and x_2 respectively) with phase -1 , arising from Figure 3.1, contributing to $Y_L^{(4)}$.

3.1.1 The observables

Definition 3.1.1. We define the operator $\hat{F}_L^{\{x_1, x_2\}}$ by marking two vertices inside the transfer matrix T_L , from Definition 1.2.3. The locations of the markers are given by x_1 and x_2 , each of which is an ordered pair (k, y) . The first coordinate, k , can take any value in $\{1, \dots, L+1\}$, corresponding to the possible positions along the width of lattice. The second coordinate y can take one of three values: b (bottom), m (middle), or t (top). The markers give weights to loops passing between them as described in the previous section.

*Throughout this chapter we take q to be the third root of unity $e^{2i\pi/3}$ and $\omega_0 = \omega_L = -4\pi/3$ as introduced in Section 1.3.

As an example of the observable we have



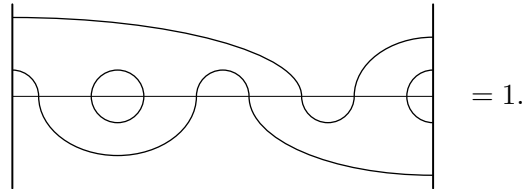
The boundary-to-boundary correlation function is given by the expectation value $F_L^{\{x_1, x_2\}}$ of this observable. We will now derive an expression for this expectation value.

The probability distribution of downward link patterns at the bottom of the transfer matrix is given by the ground state eigenvector $|\Psi(\zeta_0, \zeta_L; z_1, \dots, z_L)\rangle$, calculated in the previous chapter, which is an element of the space LP_L as defined in Section 1.1.2. To describe the probabilities for the upward link patterns at the top of the transfer matrix, we must first define a new space, $\widetilde{\text{LP}}_L$, which is spanned by the upward link patterns of length L . The elements of this space are denoted $\langle \tilde{\alpha} |$, not to be confused with the notation used for the left eigenvector of the transfer matrix defined in Section 2.1.1. For upward link patterns, we use the same shorthand notation in terms of opening and closing brackets as was defined in Section 1.1.2 for downward link patterns.

Definition 3.1.2. Between $\widetilde{\text{LP}}_L$ and LP_L we define the LP-inner product as follows. Let $|\alpha\rangle \in \text{LP}_L$ be a downward link pattern and $\langle \tilde{\alpha} | \in \widetilde{\text{LP}}_L$ be an upward link pattern. Then

$$\langle \tilde{\alpha} | \alpha \rangle_{\text{LP}} = 1, \quad \forall \alpha, \tilde{\alpha}.$$

This inner product can be viewed as the result obtained when an upward and a downward link pattern are attached to each other. All resulting closed loops and loops connected to the boundaries disappear, giving a weight of 1. This is best understood pictorially; for example, choosing $\tilde{\alpha} = '()()()('$ and $\alpha = '()()()('$, we have



The probabilities for the upward link patterns at the top of the transfer matrix are prescribed by $\langle \tilde{\Psi} |$, which is the groundstate eigenvector of the adjoint transfer matrix,

$$\langle \tilde{\Psi} | T_L := T_L^\dagger \left(\langle \tilde{\Psi} | \right) = \langle \tilde{\Psi} |.$$

Lemma 3.1.1. *The adjoint transfer matrix is given by*

$$T_L^\dagger(w; \zeta_0, \zeta_L; z_1, \dots, z_L) = T_L(qw^{-1}; \zeta_L, \zeta_0; z_L, \dots, z_1).$$

Proof. This is seen by considering the result of rotating the transfer matrix by 180° . Each vertical pair of R -matrices can be transformed by the crossing relation (1.2.11) into

The right boundary K -matrix $K_L(w)$, when rotated by 180° , is identical to $K_0(w/q)$ by (1.2.8). Similarly, $K_0(1/w)$ rotated by 180° is identical to $K_L(q/w)$. Thus, the adjoint transfer matrix can be obtained by a swapping of parameters; $\zeta_0 \leftrightarrow \zeta_L$, $z_1 \leftrightarrow z_L$, $z_2 \leftrightarrow z_{L-1}, \dots$, and replacing $w \rightarrow q/w$. \square

Lemma 3.1.2. *The elements of the vector $\langle \tilde{\Psi} |$ are related to those of $|\Psi\rangle$ by the same swapping of parameters as in Lemma 3.1.1, as well as a horizontal reflection of the link patterns, e.g., $\alpha = \text{'}((())\text{'}$ goes to $\tilde{\alpha} = \text{'()}()\text{'}$ for $L = 6$. Thus,*

$$\tilde{\psi}_\alpha(\zeta_0, \zeta_L; z_1, \dots, z_L) = \psi_\alpha(\zeta_L, \zeta_0; z_L, \dots, z_1).$$

The expectation value $F_L^{\{x_1, x_2\}}$, as defined in (3.1.1), is thus given for the infinite lattice by

$$F_L^{\{x_1, x_2\}} = \frac{\langle \tilde{\Psi} | \hat{F}_L^{\{x_1, x_2\}} | \Psi \rangle_{\text{LP}}}{\langle \tilde{\Psi} | \Psi \rangle_{\text{LP}}}, \quad (3.1.2)$$

with $\hat{F}_L^{\{x_1, x_2\}}$ as defined in Definition 3.1.1. As $\langle \tilde{\alpha} | \alpha \rangle_{\text{LP}} = 1$, $\forall \alpha, \tilde{\alpha}$, we have that

$$\langle \tilde{\Psi} | \Psi \rangle_{\text{LP}} = \sum_{\tilde{\alpha}, \alpha} \tilde{\psi}_{\tilde{\alpha}} \psi_\alpha = \tilde{Z}_L Z_L,$$

where $Z_L = \sum_\alpha \psi_\alpha$ is the normalisation of the eigenvector, given by (2.4.8). Since Z_L is symmetric in all the z_i , as well as between ζ_0 and ζ_L , we have $\tilde{Z}_L = Z_L$, so the denominator of (3.1.2) becomes Z_L^2 .

Lemma 3.1.3. *The function F_L is antisymmetric, $F_L^{\{x_1, x_2\}} = -F_L^{\{x_2, x_1\}}$ and additive, $F_L^{\{x_1, x_2\}} + F_L^{\{x_2, x_3\}} = F_L^{\{x_1, x_3\}}$.*

Proof. These properties are easily seen from the definition. \square

Because of the additivity property of F_L , it suffices to concentrate on the case where

the two markers are placed on adjacent sites, so that $N_C^{\{x_1, x_2\}}$ can only be 0 or 1. The markers can be separated by a horizontal lattice edge, or a vertical one as in Figure 3.2.

There are a number of possible positions for the pair of markers in the transfer matrix. If the markers are vertically separated, they can be situated on the top or bottom row, and at any of the $L+1$ positions along the width of the lattice. Recalling Definition 3.1.1, for the bottom row we define

$$\hat{Y}_L^{(k)} = \hat{F}_L^{\{(k,m),(k,b)\}},$$

and from the expectation value $Y_L^{(k)}$, we can calculate $F_L^{\{(k,t),(k,m)\}}$ by

$$F_L^{\{(k,t),(k,m)\}}(z_1, \dots, z_L) = Y_L^{(k)}(1/z_1, \dots, 1/z_L). \quad (3.1.3)$$

This fact comes directly from the definition of the transfer matrix. In addition, we will show (see Section 3.2.1) that Y_L is invariant under horizontal translation of the markers, so we sometimes drop the label k .

Horizontally separated markers can straddle any of the L horizontal positions, and can be in one of three vertical positions. With the markers at the very bottom we define

$$\hat{X}_L^{(k)} = \hat{F}_L^{\{(k,b),(k+1,b)\}}.$$

It will be shown in Section 3.2.2 that the expectation value $X_L^{(k)}$ is invariant under vertical translation of the markers, i.e., $F_L^{\{(k,t),(k+1,t)\}} = F_L^{\{(k,m),(k+1,m)\}} = X_L^{(k)}$. Furthermore, we can write the operator $\hat{X}_L^{(k)}$ as a product, $\hat{X}_L^{(k)} = T_L \kappa_X^{(k)}$, where κ_X is the operator consisting of the two markers. Hence,

$$X_L^{(k)} = \frac{1}{Z_L^2} \langle \tilde{\Psi} | \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} = \frac{1}{Z_L^2} \langle \tilde{\Psi} | T_L \kappa_X^{(k)} | \Psi \rangle_{\text{LP}} = \frac{1}{Z_L^2} \langle \tilde{\Psi} | \kappa_X^{(k)} | \Psi \rangle_{\text{LP}}.$$

This also implies that the expectation value $X_L^{(k)}$ does not depend on the spectral parameter w .

We will at times suppress some of the arguments of polynomials. Unless otherwise stated, $Y_L = Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L)$ with a special dependence on w , and $X_L^{(k)} = X_L^{(k)}(\zeta_0, \zeta_L; z_1, \dots, z_L)$ with a special dependence on z_k . Where relevant, variables will be given explicitly, and the notation \hat{z}_i in a list of arguments will mean that z_i is missing from the list (a convention used in the previous chapter).

3.1.2 Relations between the observables \hat{Y}_L and \hat{X}_L

Relations between \hat{X}_L and \hat{Y}_L are obtained by looking at the way the markers can move within configurations with fixed local orientations of paths.

Proposition 3.1.4. \widehat{Y}_L is related to $\widehat{X}_L^{(k)}$ by

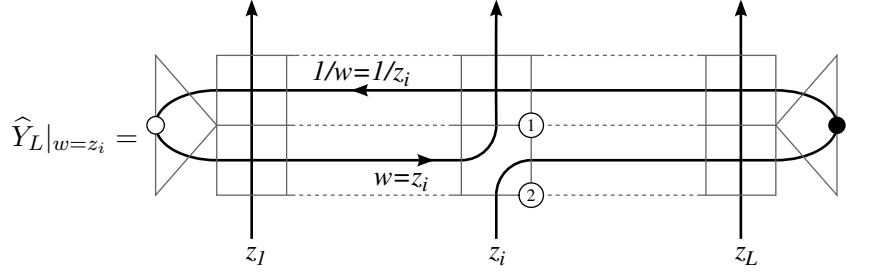
$$\widehat{Y}_L(w = z_i; z_1, \dots, z_L) = \widehat{X}_L^{(i)}(z_1, \dots, z_L),$$

and $\widehat{X}_L^{(k)}$ is related to \widehat{Y}_L by

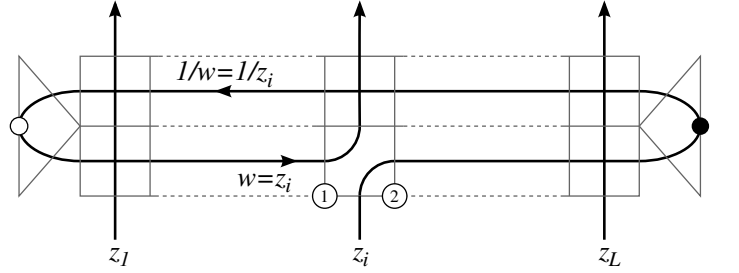
$$\begin{aligned} \widehat{X}_L^{(i+1)}(\dots, z_i = wq^{-1}, z_{i+1} = w, \dots) \circ \varphi_i &= \varphi_i \circ \widehat{Y}_{L-2}(w; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots), \\ \widehat{X}_L^{(i)}(\dots, z_i = wq^{-1}, z_{i+1} = w, \dots) \circ \varphi_i &= -\varphi_i \circ \widehat{Y}_{L-2}(w; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots), \end{aligned}$$

where φ_i is the map from Definition 2.2.2.

Proof. These relations can be proved pictorially. We first prove the relation that expresses \widehat{Y}_L in terms of \widehat{X}_L . At the specialisation $w = z_i$, the lower face at position i only has one possible configuration,



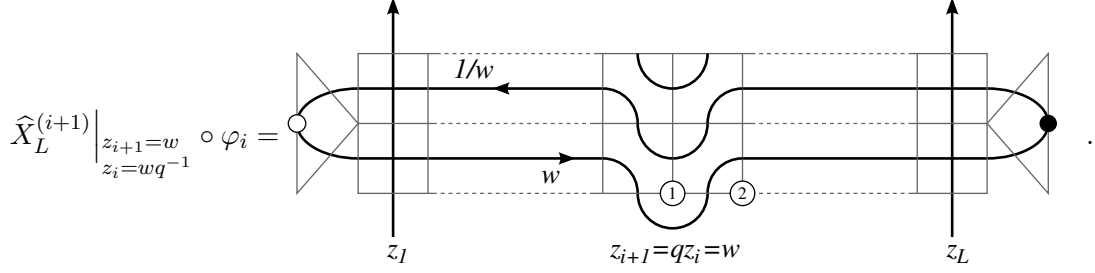
The marker x_1 can be moved along the loop on this face, producing



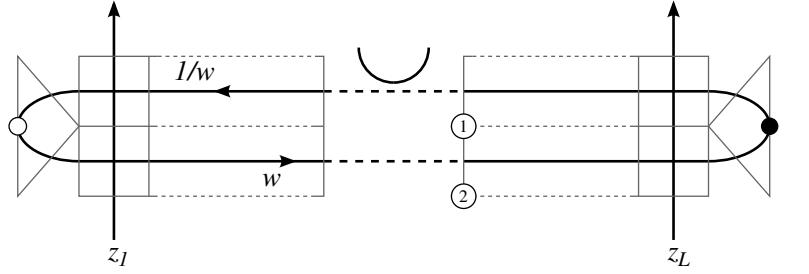
which is just $\widehat{X}_L^{(i)}$.

The relations that express \widehat{X}_L in terms of \widehat{Y}_{L-2} are similarly proved. For the first of these, the specialisation $z_{i+1} = qz_i = w$ imposes one possible configuration on the four

central faces, which acts on the small loop introduced by φ_i ,



Again the marker x_1 can be moved along the loop, and as in Proposition 2.2.1 (remembering that the chosen specialisation for q makes the proportionality factor 1), this leaves



which is exactly φ_i acting on $\hat{Y}_{L-2}(w; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots)$. The proof of the second relation is analogous. \square

3.2 Symmetries

The expectation values Y_L and $X_L^{(k)}$ have a number of symmetries, most arising from properties of the transfer matrix. In this section we will describe these symmetries.

Lemma 3.2.1. *It is easily seen that the interlacing conditions listed in (1.2.14) still hold to the left and right of the markers in the operators $\hat{X}_L^{(k)}$ or $\hat{Y}_L^{(k)}$, for example*

$$\check{R}(z_i/z_{i+1})\hat{X}_L^{(k)}(\dots, z_i, z_{i+1}, \dots) = \hat{X}_L^{(k)}(\dots, z_{i+1}, z_i, \dots)\check{R}(z_i/z_{i+1}),$$

for all values $k \neq i, i+1$.

Recall that the ground state eigenvector $|\Psi\rangle$ satisfies the q KZ equation for open boundaries (2.1.1), explicitly

$$\begin{aligned} \check{R}(z_i/z_{i+1})|\Psi(\dots, z_i, z_{i+1}, \dots)\rangle &= |\Psi(\dots, z_{i+1}, z_i, \dots)\rangle, \\ \check{K}_0(z_1^{-1}, \zeta_0)|\Psi(z_1, \dots)\rangle &= |\Psi(z_1^{-1}, \dots)\rangle, \\ \check{K}_L(z_L, \zeta_L)|\Psi(\dots, z_L)\rangle &= |\Psi(\dots, z_L^{-1})\rangle. \end{aligned} \tag{3.2.1}$$

Similarly, the vector $\langle \tilde{\Psi} |$ satisfies

$$\begin{aligned}\langle \tilde{\Psi}(\dots, z_i, z_{i+1}, \dots) | \check{R}(z_{i+1}/z_i) &= \langle \tilde{\Psi}(\dots, z_{i+1}, z_i, \dots) |, \\ \langle \tilde{\Psi}(z_1, \dots) | \check{K}_0(z_1, \zeta_0) &= \langle \tilde{\Psi}(z_1^{-1}, \dots) |, \\ \langle \tilde{\Psi}(\dots, z_L) | \check{K}_L(z_L^{-1}, \zeta_L) &= \langle \tilde{\Psi}(\dots, z_L^{-1}) |.\end{aligned}\tag{3.2.2}$$

These properties will be used to prove the two propositions below.

Proposition 3.2.2. $X_L^{(k)}$ has the following symmetries:

$$\begin{aligned}X_L^{(k)}(\dots, z_i, z_{i+1}, \dots) &= X_L^{(k)}(\dots, z_{i+1}, z_i, \dots), \quad \text{for } k \neq i, i+1, \\ X_L^{(k)}(z_1, \dots) &= X_L^{(k)}(z_1^{-1}, \dots), \quad \text{for } k \neq 1, \\ X_L^{(k)}(\dots, z_L) &= X_L^{(k)}(\dots, z_L^{-1}), \quad \text{for } k \neq L.\end{aligned}$$

Therefore, $X_L^{(k)} \in \mathbb{C}[z_1^\pm, \dots, z_{k-1}^\pm]^{\mathcal{W}_B}$, i.e., regarding the variables z_k, \dots, z_L as complex numbers, $X_L^{(k)}$ is invariant under the action of the Weyl group of type B_{k-1} and thus is symmetric in $\cup_{j=1}^{k-1} \{z_j\}$ as well as invariant under $z_j \leftrightarrow 1/z_j$, $\forall 1 \leq j \leq k-1$. Similarly, we also have that $X_L^{(k)} \in \mathbb{C}[z_{k+1}^\pm, \dots, z_L^\pm]^{\mathcal{W}_B}$.

Proof. For $k \neq i, i+1$, we use the unitarity relation (1.2.7) of the R -matrix to insert an identity into the numerator of $X_L^{(k)}$, and by the interlacing condition in Lemma 3.2.1 and the q KZ equations (3.2.1) and (3.2.2),

$$\begin{aligned}\langle \tilde{\Psi} | \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} &= \langle \tilde{\Psi} | \check{R}(z_{i+1}/z_i) \check{R}(z_i/z_{i+1}) \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} \\ &= \langle \tilde{\Psi} | \check{R}(z_{i+1}/z_i) \hat{X}_L^{(k)}(z_{i+1}, z_i) \check{R}(z_i/z_{i+1}) | \Psi \rangle_{\text{LP}} \\ &= \langle \tilde{\Psi}(z_{i+1}, z_i) | \hat{X}_L^{(k)}(z_{i+1}, z_i) | \Psi(z_{i+1}, z_i) \rangle_{\text{LP}}.\end{aligned}$$

Since the denominator of $X_L^{(k)}$ is symmetric, we have shown that $X_L^{(k)}$ is also symmetric under $z_i \leftrightarrow z_{i+1}$.

Similarly, for $k \neq 1$ and $k \neq L$ respectively,

$$\begin{aligned}\langle \tilde{\Psi} | \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} &= \langle \tilde{\Psi} | \check{K}_0(z_1, \zeta_0) \check{K}_0(z_1^{-1}, \zeta_0) \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} \\ &= \langle \tilde{\Psi}(z_1^{-1}, \dots) | \hat{X}_L^{(k)}(z_1^{-1}, \dots) \check{K}_0(z_1^{-1}, \zeta_0) | \Psi \rangle_{\text{LP}} \\ &= \langle \tilde{\Psi}(z_1^{-1}, \dots) | \hat{X}_L^{(k)}(z_1^{-1}, \dots) | \Psi(z_1^{-1}, \dots) \rangle_{\text{LP}}, \\ \langle \tilde{\Psi} | \hat{X}_L^{(k)} | \Psi \rangle_{\text{LP}} &= \langle \tilde{\Psi}(\dots, z_L^{-1}) | \hat{X}_L^{(k)}(\dots, z_L^{-1}) | \Psi(\dots, z_L^{-1}) \rangle_{\text{LP}}.\end{aligned}$$

Again, the denominator of $X_L^{(k)}$ is invariant under $z_1 \rightarrow 1/z_1$ and $z_L \rightarrow 1/z_L$, so the result is proved. \square

Proposition 3.2.3. $Y_L^{(k)}$ has the following symmetries:

$$\begin{aligned} Y_L^{(k)}(\dots, z_i, z_{i+1}, \dots) &= Y_L^{(k)}(\dots, z_{i+1}, z_i, \dots), & \text{for } k \neq i+1, \\ Y_L^{(k)}(z_1, \dots) &= Y_L^{(k)}(z_1^{-1}, \dots), & \text{for } k \neq 1, \\ Y_L^{(k)}(\dots, z_L) &= Y_L^{(k)}(\dots, z_L^{-1}), & \text{for } k \neq L+1. \end{aligned}$$

Therefore we have that $Y_L^{(k)} \in \mathbb{C}[z_1^\pm, \dots, z_{k-1}^\pm]^{\mathcal{W}_B} \cap \mathbb{C}[z_k^\pm, \dots, z_L^\pm]^{\mathcal{W}_B}$.

Proof. The proof is exactly analagous to that of Proposition 3.2.2. \square

In the next section we will deduce further symmetries of $Y_L^{(k)}$. Then in Section 3.2.2 these symmetries of Y_L will be used to find further symmetries of $X_L^{(k)}$.

3.2.1 Symmetries of Y_L

Proposition 3.2.4. $Y_L^{(k)}$ is independent of k , and is symmetric in the combined set of inhomogeneities, $Y_L^{(k)} \in \mathbb{C}[z_1^\pm, \dots, z_L^\pm]^{\mathcal{W}_B}$.

Proof. Consider an infinite lattice made up of double-row transfer matrices. Let P be the set of all paths p that start from the left boundary and end at the right. Since $Y_L^{(k)}$ depends only on the spectral parameter of one double-row transfer matrix, we can assume that the internal spectral parameters of all transfer matrices are equal. Let us call this common value w . Therefore, a path is independent of its vertical position and we shall abuse notation to also denote by p the equivalence class of all vertical translates of a particular path p . The weight ω_p of each path is the product of the Boltzmann weights associated with the local path orientations from each R and K -matrix involved in fixing p .

The set P can be split into two mutually exclusive subsets: P_t , the set of paths that start in the top half of the K -matrix on the left, and P_b , the set of paths that start in the bottom half. These sets have a 1–1 correspondence: each path $p \in P_t$ can be transformed into a path $\tilde{p} \in P_b$ by a vertical flip. The weights of the two paths are related by $\omega_p(z_1, \dots, z_L) = \omega_{\tilde{p}}(z_1^{-1}, \dots, z_L^{-1})$.

Consider now $Y_L^{(k)}$, $k \in \{1, \dots, L+1\}$, and a path $p \in P_b$. Because we must consider all of the possible vertical positions of p , it may have multiple contributions to Y_L . The number of contributions m_p depends on the number of times the path crosses back and forth over the vertical line on which the markers of \hat{Y}_L are placed. It can be seen that when k is odd, all paths in P_b contribute positively, because each path enters $\hat{Y}_L^{(k)}$ from the left. They all contribute negatively, i.e., enter from the right, when k is even. The opposite is true for the paths in P_t .

For example, for $k = 1$, the path $p_1 \in P_b$ in Figure 3.3(a) has one vertical translate that also winds through the two markers, hence its multiplicity $m_{p_1} = 2$. The sister path

\tilde{p}_1 , depicted in Figure 3.3(b), is obtained by flipping the complete configuration (but not the markers), and has $m_{\tilde{p}_1} = 1$.

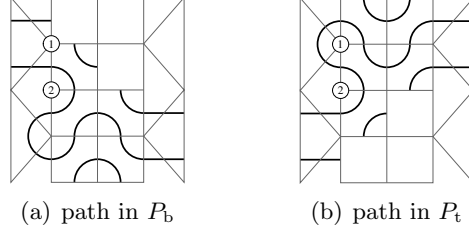


Figure 3.3: Paths contributing to $Y_2^{(1)}$.

Since a path must cross position k from left to right exactly once more than it crosses from right to left, there is always one more path contributing positively than negatively. As a consequence of this, for $p \in P_b$ and its sister path $\tilde{p} \in P_t$, it holds that $m_{\tilde{p}} = m_p - 1$ when k is odd and $m_{\tilde{p}} = m_p + 1$ when k is even. Consequently, Y_L is given by

$$\begin{aligned} Y_L^{(k)}(z_1, \dots, z_L) &= (-1)^k \left(\sum_{\tilde{p} \in P_t} m_{\tilde{p}} \omega_{\tilde{p}}(z_1, \dots, z_L) - \sum_{p \in P_b} m_p \omega_p(z_1, \dots, z_L) \right) \\ &= (-1)^k \sum_{p \in P_b} \left((m_p + (-1)^k) \omega_p(z_1^{-1}, \dots, z_L^{-1}) - m_p \omega_p(z_1, \dots, z_L) \right). \end{aligned}$$

As we showed in the previous section, $Y_L^{(k)}$ is invariant under $z_i \rightarrow 1/z_i$, for $i < k$ as well as for $i \geq k$, so consider the construction

$$\begin{aligned} Y_L^{(k)}(z_1, \dots, z_L) &= \frac{1}{2} \left(Y_L^{(k)}(z_1, \dots, z_L) + Y_L^{(k)}(z_1^{-1}, \dots, z_L^{-1}) \right) \\ &= \frac{1}{2} (-1)^k \sum_{p \in P_b} \left((m_p + (-1)^k) \omega_p(z_1^{-1}, \dots, z_L^{-1}) - m_p \omega_p(z_1, \dots, z_L) \right. \\ &\quad \left. + (m_p + (-1)^k) \omega_p(z_1, \dots, z_L) - m_p \omega_p(z_1^{-1}, \dots, z_L^{-1}) \right) \\ &= \frac{1}{2} \sum_{p \in P_b} \left(\omega_p(z_1^{-1}, \dots, z_L^{-1}) + \omega_p(z_1, \dots, z_L) \right). \end{aligned}$$

It thus follows that each translational equivalence class of paths p in P_b has only one contribution to Y_L , which is equal to the average of the weights of p and \tilde{p} . It is also clear from the above that $Y_L^{(k)}$ is in fact independent of k . Since Y_L is independent of its horizontal position, there is no longer a restriction on the symmetries, so it is symmetric in z_i and invariant under $z_j \rightarrow 1/z_j$, $\forall i, j$. \square

Proposition 3.2.5. *As a function of the spectral parameter w , Y_L is symmetric under*

$w \rightarrow q/w$ and a swapping of boundary parameters,

$$Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L) = Y_L(qw^{-1}; \zeta_L, \zeta_0; z_1, \dots, z_L).$$

Proof. From Lemma 3.1.1 we know that

$$Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L) = F_L^{\{(k,t),(k,m)\}}(qw^{-1}; \zeta_L, \zeta_0; z_L, \dots, z_1),$$

for some k , and by (3.1.3) this becomes

$$Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L) = Y_L(qw^{-1}; \zeta_L, \zeta_0; 1/z_L, \dots, 1/z_1).$$

Then use of the symmetries given in Proposition 3.2.4 completes the proof. \square

3.2.2 Symmetries of $X_L^{(k)}$

Proposition 3.2.6. $X_L^{(k)}$ is an element of $\mathbb{C}[z_1^\pm, \dots, \hat{z}_k, \dots, z_L^\pm]^{\mathcal{W}_B}$, and has the property $X_L^{(k)} = X_L^{(j)}|_{z_k \leftrightarrow z_j}$.

Proof. This comes from the symmetries of Y_L in Proposition 3.2.4, and the second relationship in Proposition 3.1.4. \square

Proposition 3.2.7. $X_L^{(k)}$ has the antisymmetry

$$X_L^{(k)}(\dots, z_k, \dots) = -X_L^{(k)}(\dots, z_k^{-1}, \dots).$$

Proof. If a path $p \in P_b$ has a contribution of $\omega_p(z_1, \dots, z_L)$ to $X_L^{(k)}$, then its sister path has the contribution $-\omega_p(z_1^{-1}, \dots, z_L^{-1})$. As a result, $X_L^{(k)}$ is antisymmetric when all z_i are sent to $1/z_i$. From the symmetries already established in Proposition 3.2.6, this antisymmetry must be attributed to $z_k \rightarrow 1/z_k$. \square

Finally, note the additivity property around an elementary plaquette,

$$\hat{X}_L^{(k)} - \hat{Y}_L^{(k+1)} - \hat{F}_L^{\{(k,m),(k+1,m)\}} + \hat{Y}_L^{(k)} = 0.$$

The invariance of Y_L implies that the operator $\hat{F}_L^{\{(k,m),(k+1,m)\}}$, with the markers in the middle of the transfer matrix, has the same expectation value as the operator $\hat{X}_L^{(k)}$.

3.3 Recursion relations

The transfer matrix allows us to deduce recursion relations that the expectation values should satisfy, and the symmetries listed in the previous section can be used to generalise these relations. In this section, we will list relations that are satisfied by the expectation

values $X_L^{(k)}$ and Y_L , in order to show that there are enough such relations, given their respective degrees, to determine these polynomials. Most of these relations are based on the recursions satisfied by the transfer matrix and the normalisation, as described in Section 2.2.

Proposition 3.3.1. *If the markers in the observable \widehat{F}_L are located sufficiently far away from the region where we specialise z_i , the following recursions hold,*

$$\begin{aligned} F_L(z_{i+1} = qz_i) &= F_{L-2}(\hat{z}_i, \hat{z}_{i+1}), \\ F_L(\zeta_0; z_1 = q\zeta_0) &= F_{L-1}(q\zeta_0; \hat{z}_1), \\ F_L(\zeta_L; z_L = q\zeta_L^{-1}) &= F_{L-1}(q\zeta_L^{-1}; \hat{z}_L). \end{aligned}$$

For Y_L , these properties always hold, as by Proposition 3.2.4 we can move the markers horizontally away from the specialised position. For $X_L^{(k)}$, the first recursion holds if $k \neq i, i+1$, and the second and third hold if $k \neq 1$ and $k \neq L$ respectively.

The proof of these recursions given in [20] did not describe the recursion of the vector $\langle \widetilde{\Psi} |$. Here the full proof is given.

Proof. We will prove here the first recursion. The other two are similarly proved, but also rely on the fact that the components of the eigenvector are invariant under $\zeta_0 \rightarrow \zeta_0^{-1}$ and $\zeta_L \rightarrow \zeta_L^{-1}$, see (2.4.1), (2.4.2), (2.4.7), and the definition of the q KZ equation in Section 2.1.

From Section 2.2.2, the eigenvector $|\Psi\rangle$ has a recursion at the point $z_{i+1} = qz_i$,

$$|\Psi_L(z_{i+1} = qz_i)\rangle = p(z_i; z_1, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots, z_L) \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle.$$

From Definition 3.1.1, it is clear that the observable \widehat{F}_L satisfies the same recursions as the transfer matrix in (2.2.3), (2.2.1) and (2.2.2),

$$\widehat{F}_L(z_{i+1} = qz_i) \circ \varphi_i = \varphi_i \circ \widehat{F}_{L-2}(\hat{z}_i, \hat{z}_{i+1}),$$

provided that the markers are not too close to i , as stated in the proposition. It remains to show

$$\langle \widetilde{\Psi}_L(z_{i+1} = qz_i) | \varphi_i \propto \langle \widetilde{\Psi}_{L-2}(\hat{z}_i, \hat{z}_{i+1}) |. \quad (3.3.1)$$

We define φ_i^\dagger to be a small upward link from position i to $i+1$, acting on a downward link pattern. We will prove here the property

$$\varphi_i^\dagger |\Psi_L(z_i = qz_{i+1})\rangle \propto |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle,$$

and then call on Lemma 3.1.2 to prove (3.3.1).

The q KZ equation (2.1.2) can be written explicitly as

$$e_i |\Psi\rangle = \sum_{\alpha} \psi_{\alpha} e_i |\alpha\rangle = \sum_{\alpha} (-a_i \psi_{\alpha}) |\alpha\rangle,$$

and from Section 2.4, we know that the only non-zero terms on the RHS are those where $\alpha = \varphi_i \beta$ for some size $L - 2$ link pattern β . Thus,

$$\sum_{\alpha} \psi_{\alpha} e_i |\alpha\rangle = \sum_{\beta} (-a_i \psi_{\varphi_i \beta}) |\varphi_i \beta\rangle,$$

where every link pattern on the RHS has a little link from i to $i + 1$.

Another way of thinking of φ_i^{\dagger} is as the bottom half of an e_i operator, which acts on $|\alpha\rangle \in \text{LP}_L$ to give $|\alpha'\rangle \in \text{LP}_{L-2}$, where $\varphi_i \alpha' = e_i \alpha$. By acting on $|\Psi\rangle$ with φ_i^{\dagger} instead of e_i , the little link from i to $i + 1$ is not introduced, so in the place of the q KZ equation we have

$$\sum_{\alpha} \psi_{\alpha} \varphi_i^{\dagger} |\alpha\rangle = \sum_{\beta} (-a_i \psi_{\varphi_i \beta}) |\beta\rangle,$$

and we want to show that when specified to $z_i = qz_{i+1}$, the RHS of this equation is proportional to the ground state eigenvector $|\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle$. Recalling the definitions of a_i in (2.1.3), we have

$$-a_i \psi_{\varphi_i \beta}^L(z_i, z_{i+1}) = \frac{[z_{i+1}/qz_i]}{[z_i/z_{i+1}]} \psi_{\varphi_i \beta}^L(z_{i+1}, z_i) + \frac{[qz_{i+1}/z_i]}{[z_i/z_{i+1}]} \psi_{\varphi_i \beta}^L(z_i, z_{i+1}).$$

Specialising $z_i = qz_{i+1}$, the second term disappears. In the first, the coefficient becomes 1 (recall $q^3 = 1$), and the first component recursion from Lemma 2.2.3 can be applied to get

$$-a_i \psi_{\varphi_i \beta}^L(z_i, z_{i+1}) \Big|_{z_i=qz_{i+1}} = p(z_{i+1}; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) \psi_{\beta}^{L-2}(\hat{z}_i, \hat{z}_{i+1}),$$

and thus,

$$\varphi_i^{\dagger} |\Psi_L(qz_{i+1}, z_{i+1})\rangle = p(z_{i+1}; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle.$$

Using Lemma 3.1.2, this result implies (3.3.1), where the proportionality factor is $p(z_i)$.

Finally, the proportionality factors obtained in the numerator and denominator of F_L cancel, resulting in the desired recursion. \square

A description of the derivation of the bulk and left boundary recursions for the transfer matrix can be found in Appendix B.

3.3.1 Relations for Y_L

Proposition 3.3.2. Y_L has the following $4L - 4$ bulk recursions,

$$Y_L \Big|_{z_i = q^{\pm 1} z_j^{\pm 1}} = Y_{L-2}(\dots, \hat{z}_i, \dots, \hat{z}_j, \dots), \quad j \neq i,$$

and for the boundaries there are 4 recursions,

$$\begin{aligned} Y_L(\zeta_0; \dots, z_i, \dots) \Big|_{z_i = (q\zeta_0)^{\pm 1}} &= Y_{L-1}((q\zeta_0)^{\pm 1}; \dots, \hat{z}_i, \dots), \\ Y_L(\zeta_L; \dots, z_i, \dots) \Big|_{z_i = (\zeta_L/q)^{\pm 1}} &= Y_{L-1}((\zeta_L/q)^{\pm 1}; \dots, \hat{z}_i, \dots). \end{aligned}$$

Proof. By the symmetries of Y_L in Proposition 3.2.4, the recursions in Proposition 3.3.1 can be generalised; for instance, the bulk recursions are true with $z_i = qz_j$, not just $z_i = qz_{i-1}$. \square

Proposition 3.3.3. We also have the relations

$$\begin{aligned} Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L) \Big|_{w = z_i^{\pm 1}} &= \pm X_L^{(i)}(\zeta_0, \zeta_L; z_1, \dots, z_L), \\ Y_L(w; \zeta_0, \zeta_L; z_1, \dots, z_L) \Big|_{w = qz_i^{\pm 1}} &= \mp X_L^{(i)}(\zeta_L, \zeta_0; z_1, \dots, z_L). \end{aligned}$$

Proof. These come from Proposition 3.1.4, the antisymmetry of $X_L^{(k)}$ in Proposition 3.2.7 and the symmetry of Y_L in Proposition 3.2.5. \square

Viewing the above relations as specialisations for z_i , and assuming that we know X_L , Y_{L-1} and Y_{L-2} , we thus find from Proposition 3.3.2 and Proposition 3.3.3 the value of Y_L in $4L + 4$ values of z_i .

3.3.2 Relations for $X_L^{(k)}$

Proposition 3.3.4. $X_L^{(k)}$ satisfies the $4L - 8$ bulk recursions (we have taken $i > k$ without loss of generality),

$$\begin{aligned} X_L^{(k)} \Big|_{z_i = q^{\pm 1} z_j^{\pm 1}} &= X_{L-2}^{(k)}(\dots, \hat{z}_i, \dots, \hat{z}_j, \dots), \quad j > k, \\ X_L^{(k)} \Big|_{z_i = q^{\pm 1} z_j^{\pm 1}} &= X_{L-2}^{(k-1)}(\dots, \hat{z}_j, \dots, \hat{z}_i, \dots), \quad j < k, \end{aligned}$$

and 4 recursions for the boundaries,

$$\begin{aligned} X_L^{(k)}(\zeta_0; \dots, z_i, \dots) \Big|_{z_i = (q\zeta_0)^{\pm 1}} &= X_{L-1}^{(k)}((q\zeta_0)^{\pm 1}; \dots, \hat{z}_i, \dots), \\ X_L^{(k)}(\zeta_L; \dots, z_i, \dots) \Big|_{z_i = (\zeta_L/q)^{\pm 1}} &= X_{L-1}^{(k)}((\zeta_L/q)^{\pm 1}; \dots, \hat{z}_i, \dots). \end{aligned}$$

Proof. Again, the recursions in Proposition 3.3.1 can be generalised by the symmetries of $X_L^{(k)}$ in Proposition 3.2.6, provided $i, j \neq k$. \square

Proposition 3.3.5. $X_L^{(k)}$ also satisfies the relations

$$\begin{aligned} X_L^{(k)} \Big|_{z_i=(z_k/q)^{\pm 1}} &= Y_{L-2}(z_k; \hat{z}_i, \hat{z}_k), \\ X_L^{(k)} \Big|_{z_i=(qz_k)^{\pm 1}} &= -Y_{L-2}(z_k^{-1}; \hat{z}_i, \hat{z}_k), \end{aligned}$$

provided $k \neq i$.

Proof. These come from Proposition 3.1.4, and the symmetry of Y_L under $w \rightarrow q/w$. \square

Therefore, assuming we know Y_{L-2} , $X_{L-2}^{(k)}$ and $X_{L-1}^{(k)}$, we know the value of $X_L^{(k)}$, viewed as a polynomial in z_i , in $4L$ points.

3.4 Degrees of \hat{Y}_L and \hat{X}_L

We recall definition (3.1.2) of the expectation value

$$F_L^{\{x_1, x_2\}} = \frac{1}{Z_L^2} \langle \tilde{\Psi} | \hat{F}_L^{\{x_1, x_2\}} | \Psi \rangle_{\text{LP}}.$$

Conjecture 3.4.1. The numerator of F_L contains a factor Z_L , cancelling one of such factors in the denominator. Equivalently, F_L is of the form P/Z_L , where P is a polynomial.

For small systems we have observed this to be true, but it is still an outstanding problem for general L .

Proposition 3.4.1. *Conditioned on Conjecture 3.4.1, the numerator of $X_L^{(k)}$ must be of degree width $4L - 2$ in each variable, and the numerator of Y_L must be of degree width $4L + 2$. Thus we have enough recursion relations, listed in the previous section, to fix the numerators of both expectation values.*

Proof. For both expectation values $X_L^{(k)}$ and Y_L , the degree width of the numerator must equal the degree width of the denominator. The degree width of Z_L is $4L - 2$ in each variable z_i . By the above conjecture, this means for $X_L^{(k)}$ that the numerator must also be of degree width $4L - 2$ in each variable.

For Y_L , we must also take into account the action of the transfer matrix. The numerator and denominator of each term in the transfer matrix have degree width 4 in each variable. We can factor the denominator out, so the degree width of both the numerator and the denominator of Y_L are now $4L + 2$. \square

3.5 Solution

Before we present the exact expressions for $X_L^{(k)}$ and Y_L , we must introduce some auxiliary functions.

Definition 3.5.1. With a slight change in notation from (2.3.8), let

$$\tau_L(z_1, \dots, z_L) = \chi_{\lambda^{(L)}}(z_1^2, \dots, z_L^2), \quad \lambda_j^{(L)} = \left\lfloor \frac{L-j}{2} \right\rfloor,$$

where the symplectic character χ_λ is defined in (2.3.7).

Definition 3.5.2. We also define

$$u_L(\zeta_0, \zeta_L; z_1, \dots, z_L) = c_L \log \left[\frac{\tau_{L+1}(\zeta_0, z_1, \dots, z_L) \tau_{L+1}(\zeta_L, z_1, \dots, z_L)}{\tau_L(z_1, \dots, z_L) \tau_{L+2}(\zeta_0, \zeta_L, z_1, \dots, z_L)} \right],$$

where the constant c_L is given by

$$c_L = (-1)^L i \frac{\sqrt{3}}{2}.$$

It is interesting to note that the function u_L has the form of a Toda lattice wave function; we refer the interested reader to [47].

Theorem 3.5.1. *Conditioned on Conjecture 3.4.1 and Conjecture 3.5.1 (below), the expectation values $X_L^{(k)}$ and Y_L are given by*

$$X_L^{(k)} = z_k \frac{\partial}{\partial z_k} u_L(\zeta_0, \zeta_L; z_1, \dots, z_L), \quad (3.5.1)$$

$$Y_L = w \frac{\partial}{\partial w} u_{L+2}(\zeta_0, \zeta_L; z_1, \dots, z_L, vq^{-1}, w) \Big|_{v=w}, \quad (3.5.2)$$

where v is a dummy variable.

This will be proved in the next section.

3.5.1 Proof of the main result

It is easy to see that the degree of the numerators of (3.5.1) and (3.5.2) are $4L - 2$ and $4L + 2$ respectively, as required by Proposition 3.4.1. In addition, the denominator of (3.5.1) is Z_L , and the denominator of (3.5.2) is Z_L times the denominator of the transfer matrix. We have also calculated by brute force the expectation values for system sizes $L = 1, 2$ and 3 and verified the solution for these cases. Thus, in order to prove that these expressions are the correct forms for $X_L^{(k)}$ and Y_L in general, we must prove that they satisfy all the required recursions as set out in Section 3.3.

Lemma 3.5.2. *The function u has a list of simple recursions, based on the recursion for τ in (2.3.9):*

$$\begin{aligned} u_L(\zeta_0, \zeta_L; \dots, z_{L-2}, z, qz) &= u_{L-2}(\zeta_0, \zeta_L; \dots, z_{L-2}), \\ u_L(\zeta_0, \zeta_L; z_1 = q\zeta_0, z_2, \dots) &= -u_{L-1}(q\zeta_0, \zeta_L; z_2, \dots) - c_L \log [-k(\zeta_0, \zeta_L)], \\ u_L(\zeta_0, \zeta_L; \dots, z_{L-1}, z_L = q^{-1}\zeta_L) &= -u_{L-1}(\zeta_0, q^{-1}\zeta_L; \dots, z_{L-1}) - c_L \log [-k(q^{-1}\zeta_L, \zeta_0)]. \end{aligned}$$

Proof. The first of these is easily obtained by using the recursion (2.3.9) on each of the τ -functions in the definition of u_L , and noting that the proportionality factors on the top and bottom cancel exactly. The second recursion we prove now, and the third works in a similar way:

$$\begin{aligned} u_L(z_1 = q\zeta_0) &= c_L \log \left[\frac{\tau_{L+1}(\zeta_0, q\zeta_0, z_2, \dots, z_L)}{\tau_L(q\zeta_0, z_2, \dots, z_L)} \frac{\tau_{L+1}(\zeta_L, q\zeta_0, z_2, \dots, z_L)}{\tau_{L+2}(\zeta_0, \zeta_L, q\zeta_0, z_2, \dots, z_L)} \right] \\ &= c_L \log \left[\frac{-1}{k(\zeta_0, \zeta_L)} \frac{\tau_{L-1}(z_2, \dots, z_L)}{\tau_L(q\zeta_0, z_2, \dots, z_L)} \frac{\tau_{L+1}(q\zeta_0, \zeta_L, z_2, \dots, z_L)}{\tau_L(\zeta_L, z_2, \dots, z_L)} \right] \\ &= -c_L \log \left[\frac{\tau_L(q\zeta_0, z_2, \dots, z_L)}{\tau_{L-1}(z_2, \dots, z_L)} \frac{\tau_L(\zeta_L, z_2, \dots, z_L)}{\tau_{L+1}(q\zeta_0, \zeta_L, z_2, \dots, z_L)} \right] - c_L \log [-k(\zeta_0, \zeta_L)] \\ &= -u_{L-1}(q\zeta_0, \zeta_L; z_2, \dots) - c_L \log [-k(\zeta_0, \zeta_L)]. \quad \square \end{aligned}$$

These recursions immediately imply the recursions for Y_L and $X_L^{(k)}$ that do not involve the marked parameter; that is, the $4L$ recursions in Proposition 3.3.2 for Y_L , as well as the $4L - 4$ recursions in Proposition 3.3.4 for X_L . The remaining recursions, from Proposition 3.3.3 and Proposition 3.3.5, which respectively complete the degree arguments for Y_L and $X_L^{(k)}$, are more complicated. These two recursions are addressed in the next section.

Relations between Y_L and $X_L^{(k)}$ in Proposition 3.3.3 and Proposition 3.3.5

Conjecture 3.5.1. The proposed expression for Y_L in (3.5.2) is invariant under the transformation $w \rightarrow q/w$.

The proof of this in [20] was found to be flawed, and a correct proof has not yet been found, so we merely assume this symmetry. It can be shown explicitly that Y_L satisfies this symmetry for small L (we have confirmed it for $L = 2$ and 3), and we expect this symmetry to hold for arbitrary L .

Proposition 3.5.3. *The expressions (3.5.1) and (3.5.2) satisfy the first relation in Proposition 3.3.3, where $w = z_i^{\pm 1}$. Conditioned on Conjecture 3.5.1 they also satisfy the second relation, where $w = qz_i^{\pm 1}$.*

Proof. Taking expression (3.5.2) for Y_L at the point $w = z_i$, and using the first recursion

in Lemma 3.5.2,

$$\begin{aligned}
& w \frac{\partial}{\partial w} u_{L+2}(z_1, \dots, z_i, \dots, z_L, qv^{-1}, w) \Big|_{\substack{v=z_i \\ w=z_i}} \\
&= w \frac{\partial}{\partial w} u_L(z_1, \dots, \hat{z}_i, \dots, z_L, w) \Big|_{w=z_i} \\
&= z_i \frac{\partial}{\partial z_i} u_L(z_1, \dots, z_L),
\end{aligned}$$

which is exactly $X_L^{(i)}$ from (3.5.1). The other relations in Proposition 3.3.3 follow by the antisymmetry of (3.5.1) under $z_i \rightarrow 1/z_i$ and Conjecture 3.5.1. \square

Proposition 3.5.4. *The expressions (3.5.1) and (3.5.2) satisfy the relations listed in Proposition 3.3.5.*

Proof. We will prove the cases where $z_i = z_k q^{-1}$ and $z_i = (qz_k)^{-1}$, and the other two cases follow from symmetry under $z_i \rightarrow z_i^{-1}$. We take the expression for $X_L^{(k)}$ (3.5.1) at the point $z_i = z_k q^{-1}$,

$$\begin{aligned}
& z_k \frac{\partial}{\partial z_k} u_L(\dots, z_i, \dots, z_k, \dots) \Big|_{z_i=z_k q^{-1}} \\
&= z_k \frac{\partial}{\partial z_k} u_L(z_1, \dots, \hat{z}_i, \dots, \hat{z}_k, \dots, z_L, vq^{-1}, z_k) \Big|_{v=z_k},
\end{aligned}$$

which is exactly the expression for Y_L in (3.5.2) at the point $w = z_k$. Similarly, at $z_i = (qz_k)^{-1}$, (3.5.1) becomes

$$\begin{aligned}
& z_k \frac{\partial}{\partial z_k} u_L(\dots, z_i, \dots, z_k, \dots) \Big|_{z_i=(qz_k)^{-1}} \\
&= -z_k^{-1} \frac{\partial}{\partial(z_k^{-1})} u_L(z_1, \dots, \hat{z}_i, \dots, \hat{z}_k, \dots, z_L, vq^{-1}, z_k^{-1}) \Big|_{v=z_k^{-1}},
\end{aligned}$$

which is exactly (3.5.2) at the point $w = z_k^{-1}$. \square

We expect both Conjecture 3.4.1 and Conjecture 3.5.1 to hold true for arbitrary system size. Under these conditions Theorem 3.5.1 is proved.

Chapter 4

Separation of variables for the symplectic character

In this thesis we have found exact finite size expressions for two quantities related to the two-boundary $TL(1)$ loop model: the normalisation Z of the ground state eigenvector, see (2.4.8); and the boundary-to-boundary correlation function F , see (3.5.1) and (3.5.2). Both of these quantities are expressed in terms of the polynomial character of the symplectic group, χ , defined in (2.3.7).

For each version of the $TL(1)$ loop model with left and right boundary conditions, physical quantities can be expressed in terms of this character. The function χ depends on the bulk inhomogeneity parameters as well as, if applicable, the parameters corresponding to the boundaries of the lattice. Analogous calculations in the periodic version of the $TL(1)$ loop model result in physical quantities that can be expressed in terms of the Schur function, a multivariate polynomial well-known in representation theory.

The Schur function, corresponding to the root system of type A (in the classification of Dynkin diagrams), is symmetric in all its variables, and it is the polynomial character of the general linear group. The symplectic character is an analogue of the Schur function for the root system of type C ; that is, not only is it a symmetric function, but it is also invariant under inversion of its arguments. The boundary loop models have this symmetry because of the effect of the boundaries, which do not exist in the periodic case.

In addition to appearing in solutions of statistical mechanical models, both the symplectic character and the Schur function play an important role in the representation theory of the classical groups [42]. They are also used as generating functions for counting problems in enumerative combinatorics; for example, each one provides a generalised enumeration of alternating sign matrices with various boundary conditions, as well as plane partitions and rhombus tilings [15, 36, 71].

For all of these applications it is important to understand the asymptotic behaviour of the polynomials, as the number of variables goes to infinity. In the case of the two-

boundary TL(1) loop model, the calculation of the asymptotic limit of the symplectic character will give the thermodynamic limit of the physical quantities Z and F . In particular we would like to analyse the effect of the two boundaries in the asymptotic limit, which involves setting the bulk parameters to 1 and keeping the boundary parameters general.

Okounkov and Olshanski studied the asymptotic limit of type A Jack polynomials and type BC orthogonal polynomials with all but a finite subset of variables set to 1 [72, 73]. For both of these problems, the authors considered polynomials with a degree growing linearly with the number of variables. In contrast, the symplectic character that we would like to analyse has a degree that is quadratic in the number of variables (see (2.4.5)) and, as far as we are aware, its asymptotic behaviour is an open problem.

With the asymptotic limit described above as our motivation, in this chapter we study a method for the separation of variables (SoV) of symplectic characters, initiated by Sklyanin [92] and extended by Kuznetsov, Mangazeev and Sklyanin [56–59]. This SoV method makes use of the Q -operator introduced by Baxter [4] for the solution of the 8-vertex model. The aim of the so-called Q -operator method is to transform a chosen multi-variable polynomial into a product of single variable polynomials. Using this method, the problem of the asymptotics will be reduced to that of finding the asymptotics for each factorised part. In order for this approach to be useful the Q -operator method must also be reversible, which is one of the main technical hurdles in SoV.

Symmetric polynomials are eigenfunctions of certain multivariate differential operators, or Hamiltonians $\{H_i\}$, which form a quantum integrable system. The Q -operator for a quantum integrable system is a quantisation of the Bäcklund transformation for the corresponding classical integrable system. This connection was first found by Pasquier and Gaudin [76], who discovered a correspondence between the classical Bäcklund transformation and the Q -operator for the periodic Toda lattice. The operator Q can be realised as an integral operator with a simple kernel. In the quasi-classical limit, this kernel turns into the generating function of the canonical transform [58, 76].

A number of examples are given in [59] for the application of the Q -operator method to systems of symmetric polynomials associated with the root system of type A . In particular, [59] discusses SoV for Schur functions, being a special case of Jack polynomials. As indicated above, this chapter aims to extend SoV using the Q -operator method to the irreducible character of the symplectic group, χ_λ , which is the Schur polynomial for the root system of type C . The only other study of SoV methods for root systems other than type A that we are aware of is for the open quantum $\mathfrak{sl}(2, \mathbb{R})$ spin chain [25].

The key hurdle faced when applying SoV to type A polynomials is the inversion of the method. The inverse of the separating operator has not been found for type A polynomials in general (a notable exception is the Schur function, see [59] for more details), and another method has to be used to reverse the process, known as the ‘lifting

operator'. For the symplectic character, it has not been possible to find an equivalent lifting operator, but the inverse separating operator is easy to find, presumably because of the symplectic character's relationship to the Schur function. This raises some questions about the invertibility of the method for more general type C polynomials.

The key ingredient of the Q -operator method is the construction of the separating operator \mathcal{S} whose action on a polynomial P_λ is proportional to a product of single variable polynomials,

$$(\mathcal{S}P_\lambda)(x_1, \dots, x_L) = P_\lambda(1, \dots, 1) \prod_{i=1}^L q_\lambda(x_i). \quad (4.0.1)$$

For the case of the symplectic character, as with Schur functions, \mathcal{S} is invertible. The structure of the method is given in the next section.

4.1 The Q -operator method

We consider a quantum integrable system with a commuting set of Hamiltonians, given by differential operators H_i ($i \in \{1, \dots, L\}$). These have eigenstates equal to the polynomial P_λ :

$$H_j P_\lambda(x_1, \dots, x_L) = h_j(\lambda) P_\lambda(x_1, \dots, x_L), \quad (4.1.1)$$

where the h_j are the corresponding eigenvalues, which depend on the multi-degree $\lambda = (\lambda_1, \dots, \lambda_L)$ of the polynomial.

The aim of the Q -operator method is to find a version of the spectral problem (4.1.1) that involves a related factorised polynomial in place of P_λ . Acting on both sides of (4.1.1) with \mathcal{S} ,

$$\mathcal{S}H_j\mathcal{S}^{-1} (\mathcal{S}P_\lambda)(x_1, \dots, x_L) = h_j(\lambda) (\mathcal{S}P_\lambda)(x_1, \dots, x_L),$$

and using (4.0.1),

$$(\mathcal{S}H_j\mathcal{S}^{-1}) \prod_{i=1}^L q_\lambda(x_i) = h_j(\lambda) \prod_{i=1}^L q_\lambda(x_i).$$

The main results of this chapter are the explicit construction of the operators \mathcal{S} and \mathcal{S}^{-1} for the symplectic characters, as well as a 'factorised Hamiltonian' \tilde{H}_j , which acts in the same way as $\mathcal{S}H_j\mathcal{S}^{-1}$ on the factorised polynomial. \tilde{H}_j is not uniquely defined, and can be constructed using q_λ as shown in the next section.

4.1.1 Factorised Hamiltonian

The operator \tilde{H}_j can be constructed from a single variable operator in the following way. If there exists a differential operator in x_i , denoted W_i , such that

$$W_i q_\lambda(x_i) = 0, \quad [W_i, q_\lambda(x_k)] = 0 \quad (k \neq i), \quad (4.1.2)$$

then we construct the operator

$$W_{i,j} = W_i + h_j(\lambda).$$

Noting that $W_{i,j}$ commutes with any function of x_k where $k \neq i$, it is easy to see that

$$W_{i,j} \prod_{k=1}^L q_\lambda(x_k) = h_j(\lambda) \prod_{k=1}^L q_\lambda(x_k), \quad \forall i, \quad (4.1.3)$$

and therefore any linear combination of the form

$$\tilde{H}_j = \sum_{i=1}^L c_i W_{i,j}, \quad \sum_{i=1}^L c_i = 1,$$

will also satisfy (4.1.3).

4.1.2 Factorisation of the separating operator \mathcal{S}

We will show that \mathcal{S} can be written as a product of operators,

$$\mathcal{S} = (\rho_0 Q_{z_1} \cdots Q_{z_L}) \big|_{z_1=x_1, \dots, z_L=x_L}, \quad (4.1.4)$$

where the operator Q_{z_i} is an integral operator that acts as

$$(Q_{z_i} P_\lambda)(x_1, \dots, x_L) = q_\lambda(z_i) P_\lambda(x_1, \dots, x_L), \quad (4.1.5)$$

and ρ_0 sends $f(x_1, \dots, x_L) \rightarrow f(1, \dots, 1)$. Furthermore, the Q_{z_i} have the important properties

$$[Q_{z_i}, Q_{z_j}] = 0, \quad [Q_{z_i}, H_j] = 0, \quad \forall i, j.$$

Having found the operator Q_{z_i} , we introduce new operators \mathcal{A}_i for which

$$(\rho_{i-1} Q_{z_i}) \big|_{z_i=x_i} = \mathcal{A}_i \rho_i.$$

Here, ρ_j sends $f(x_1, \dots, x_L) \rightarrow f(x_1, \dots, x_j, 1, \dots, 1)$. These new operators act as

$$(\mathcal{A}_i P_\lambda)(x_1, \dots, x_i, 1, \dots, 1) = q_\lambda(x_i) P_\lambda(x_1, \dots, x_{i-1}, 1, \dots, 1), \quad (4.1.6)$$

and (4.1.4) becomes the factorisation

$$\mathcal{S} = \mathcal{A}_1 \dots \mathcal{A}_L.$$

4.1.3 Summary

Given a particular family of polynomials, the Q -operator method can be condensed into five steps:

1. Specify H_j and $h_j(\lambda)$ in (4.1.1),
2. Specify W_i satisfying (4.1.2), and construct the factorised Hamiltonian,
3. Construct \mathcal{S}^{-1} ,
4. Find Q_z such that (4.1.5) is satisfied,
5. Find \mathcal{A}_i such that (4.1.6) is satisfied, and construct \mathcal{S} .

The inverse separating operator is given before the construction of \mathcal{S} for two reasons; firstly, for the symplectic character it turns out to be simpler in form, and secondly, the details relate to those of the differential operators H and W used in the first two steps.

As previously stated, this chapter focuses on the case where P_λ is equal to χ_λ , the symplectic character, which will be defined in Section 4.2. For this case, we closely follow [59], in which the factorisation of the Schur functions is treated. The technical detail in the symplectic case however is more elaborate.

Section 4.2 describes the symplectic character and some of its properties. The sections following proceed in the order of the steps above. Section 4.5 also describes the inverse of the operator $\mathcal{S}_k = \mathcal{A}_1 \dots \mathcal{A}_k$, which satisfies

$$\mathcal{S}_k^{-1} \prod_{i=1}^k q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \dots, x_k, 1, \dots, 1)}{\chi_\lambda(1, \dots, 1)}.$$

4.2 Properties of the symplectic character

The polynomials we will consider, defined in (2.3.7), are the irreducible characters of the symplectic group. These form a basis of $\mathbb{C}[x_1^\pm, \dots, x_L^\pm]^{\mathcal{W}_B}$, the ring of Laurent polynomials with complex coefficients invariant under the action of the Weyl group of type B_L , i.e., polynomials symmetric in x_i and invariant under $x_i \rightarrow x_i^{-1}$. Each character is labelled by a partition $\lambda = (\lambda_1, \dots, \lambda_L)$, with $\lambda_i \geq \lambda_{i+1}$. We refer to them as symplectic characters [42, 103], and for the purposes of this chapter define them as follows,

Definition 4.2.1.

$$\chi_\lambda(x_1, \dots, x_L) = \frac{a_\mu(x_1, \dots, x_L)}{a_\delta(x_1, \dots, x_L)},$$

where

$$a_\mu(x_1, \dots, x_L) = \det_L \left[x_i^{\mu_j} - x_i^{-\mu_j} \right],$$

δ is defined as the partition $(L, \dots, L-i+1, \dots, 2, 1)$, and we use μ to denote the partition $\lambda + \delta$, so $\mu_i = \lambda_i + L - i + 1$. The subscript L on the determinant denotes that the matrix is size $L \times L$.

The polynomials a_μ are Laurent polynomials in the ring $\mathbb{C}[x_1^\pm, \dots, x_L^\pm]$ that are antisymmetric in (x_1, \dots, x_L) and under $x_i \rightarrow x_i^{-1}$. These antisymmetries mean that $a_\mu(x_1, \dots, x_L) = 0$ whenever $x_i = x_j$ or $x_i = 1$. For the definition of \mathcal{A}_i , we will need an expression for χ_λ when some of the arguments are set to 1, so we must find an alternate definition for χ_λ in this limit. We therefore define a ‘truncated’ version of a_μ :

Definition 4.2.2.

$$a_\mu^{(k)}(x_1, \dots, x_{k-1}) = \det_L \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{i < k} \\ [\mu_j^{2(L-i)+1}]_{i \geq k} \end{bmatrix}. \quad (4.2.1)$$

Lemma 4.2.1. *For $i = k, \dots, L$, set $x_i = e^{\varepsilon u_i}$ with $u_i \in \mathbb{R}$. Then*

$$a_\mu^{(k)}(x_1, \dots, x_{k-1}) \sim c_k(\varepsilon) a_\mu(x_1, \dots, x_L),$$

in the limit as $\varepsilon \rightarrow 0$. The prefactor $c_k(\varepsilon)$ does not depend on μ .

Proof. As an example, we first prove the above for the case where $k = L$. In this case the parameter u can be set to 1. In the L th row of the determinant $a_\mu(x_1, \dots, x_{L-1}, e^\varepsilon)$, the j th element is

$$\begin{aligned} e^{\varepsilon \mu_j} - e^{-\varepsilon \mu_j} &= 2 \sinh(\varepsilon \mu_j) \\ &= 2\varepsilon \mu_j + \text{h.o.t.} \end{aligned}$$

Taking this to first order in ε , we can factor 2ε out of the determinant, leaving μ_j in the bottom row. Then

$$a_\mu^{(L)}(x_1, \dots, x_{L-1}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} a_\mu(x_1, \dots, x_{L-1}, e^\varepsilon),$$

so $c_L(\varepsilon) = 1/2\varepsilon$.

In the case of general k , the parameters u_i allow us to take multiple arguments to 1 along distinct trajectories. In the rows from k to L of $a_\mu(x_1, \dots, x_{k-1}, e^{\varepsilon u_k}, \dots, e^{\varepsilon u_L})$, we expand up to order $\varepsilon^{2(L-k)+1}$,

$$2 \sinh(\varepsilon u_i \mu_j) \approx \sum_{m=0}^{L-k} \frac{2(\varepsilon u_i \mu_j)^{2m+1}}{(2m+1)!} = \sum_{m=k}^L \frac{2(\varepsilon u_i \mu_j)^{2(L-m)+1}}{(2(L-m)+1)!}.$$

The resulting matrix can be shown to be a product of two matrices. We use $\mathbf{1}_n$ for the

identity matrix of size n , and $\mathbf{0}$ for a matrix of zeroes whose size is determined from context:

$$\begin{aligned}
& \det_L \begin{bmatrix} \left[x_i^{\mu_j} - x_i^{-\mu_j} \right]_{i < k} \\ \left[\sum_{m=k}^L \frac{2(\varepsilon u_i \mu_j)^{2(L-m)+1}}{(2(L-m)+1)!} \right]_{i \geq k} \end{bmatrix} \\
&= \det_L \left(\begin{bmatrix} \mathbf{1}_{k-1} & \mathbf{0} \\ \mathbf{0} & \left[\frac{2(\varepsilon u_i)^{2(L-j)+1}}{(2(L-j)+1)!} \right]_{i,j \geq k} \end{bmatrix} \begin{bmatrix} \left[x_i^{\mu_j} - x_i^{-\mu_j} \right]_{i < k} \\ \left[\mu_j^{2(L-i)+1} \right]_{i \geq k} \end{bmatrix} \right) \\
&= \frac{2^{L-k+1} \varepsilon^{(L-k+1)^2}}{\prod_{m=k}^L (2(L-m)+1)!} a_{(u_k, \dots, u_L)}^{(1)} a_{\mu}^{(k)}(x_1, \dots, x_{k-1}).
\end{aligned}$$

This leads to the result

$$\begin{aligned}
& a_{\mu}^{(k)}(x_1, \dots, x_{k-1}) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-(L-k+1)^2} \prod_{m=0}^{L-k} (2m+1)!}{2^{L-k+1} a_{(u_k, \dots, u_L)}^{(1)}} a_{\mu}(x_1, \dots, x_{k-1}, e^{\varepsilon u_k}, \dots, e^{\varepsilon u_L}),
\end{aligned} \tag{4.2.2}$$

so the proportionality factor $c_k(\varepsilon)$ is

$$c_k(\varepsilon) = \frac{\varepsilon^{-(L-k+1)^2} \prod_{m=0}^{L-k} (2m+1)!}{2^{L-k+1} a_{(u_k, \dots, u_L)}^{(1)}}. \quad \square$$

We note that there is an alternative inductive proof similar to that used in [59], which we give in Appendix E.

Corollary. *Because $c_k(\varepsilon)$ is independent of μ , at the point $x_{k+1} = \dots = x_L = 1$ we have an alternate definition of χ ,*

$$\chi_{\lambda}(x_1, \dots, x_k, 1, \dots, 1) = \frac{a_{\mu}^{(k+1)}(x_1, \dots, x_k)}{a_{\delta}^{(k+1)}(x_1, \dots, x_k)}.$$

This new definition allows us to directly calculate the symplectic character when some of its arguments are set to 1.

4.2.1 Factorised forms and homogeneous identities

This section will list some useful identities and factorised expressions for a_{μ} and χ_{λ} . First recall that the denominator of a Schur function is given by the Vandermonde determinant,

which has the product form

$$\det_L \left[x_i^{L-j} \right] = \prod_{1 \leq m < n \leq L} (x_m - x_n). \quad (4.2.3)$$

The Weyl denominator formula for type C gives us an analogous identity for the symplectic character [42],

$$a_\delta(x_1, \dots, x_L) = \prod_{i=1}^L x_i^{i-L} \prod_{1 \leq i < j \leq L} (x_i - x_j) \prod_{1 \leq i < j \leq L} (x_i - x_j^{-1}). \quad (4.2.4)$$

We can also use (4.2.3) to show

$$a_\mu^{(1)} = \det_L \left[\mu_i^{2(L-j)+1} \right] = \prod_{i=1}^L \mu_i \prod_{1 \leq i < j \leq L} (\mu_i^2 - \mu_j^2). \quad (4.2.5)$$

This result and simple row expansion of (4.2.1) immediately leads to the following identity,

$$a_\mu^{(2)}(z) = \sum_{k=1}^L (-1)^{k-1} \left(\prod_{i \neq k} \mu_i \prod_{\substack{1 \leq i < j \leq L \\ i, j \neq k}} (\mu_i^2 - \mu_j^2) \right) (z^{\mu_k} - z^{-\mu_k}), \quad (4.2.6)$$

which will be useful later. Furthermore, from (4.2.4) and (4.2.2), it is easily shown that

$$\begin{aligned} a_\delta^{(k)}(x_1, \dots, x_{k-1}) &= \prod_{i=0}^{L-k} (2i+1)! \prod_{i=1}^{k-1} x_i^{i-L} (x_i - 1)^{2(L-k+1)} \\ &\times \prod_{1 \leq i < j \leq k-1} (x_i - x_j) \prod_{1 \leq i < j \leq k-1} (x_i - x_j^{-1}). \end{aligned} \quad (4.2.7)$$

In particular,

$$a_\delta^{(1)} = \prod_{i=1}^{L-1} (2i+1)!, \quad (4.2.8)$$

which, along with (4.2.5), leads to

$$\chi_\lambda(1, \dots, 1) = \prod_{i=1}^L \frac{\mu_i}{(2i-1)!} \prod_{1 \leq i < j \leq L} (\mu_i^2 - \mu_j^2),$$

which is Weyl's dimension formula for the symplectic group [42, 103].

4.3 Hamiltonians and eigenvalues

We now construct the system of mutually commuting Hamiltonians H_j that satisfy the eigenvalue equation (4.1.1) in the symplectic case, i.e.,

$$H_j \chi_\lambda(\mathbf{x}) = h_j(\lambda) \chi_\lambda(\mathbf{x}), \quad (4.3.1)$$

where we have used a bold \mathbf{x} to refer to a list of variables x_1, \dots, x_L .

Let $D_x = x \frac{\partial}{\partial x}$, and recall the definition of the usual elementary symmetric function e_j ,

$$e_j(x_1, \dots, x_L) = \sum_{1 \leq r_1 < \dots < r_j \leq L} x_{r_1} \dots x_{r_j}.$$

Lemma 4.3.1. *The Hamiltonians H_j , for $j = 1, \dots, L$, are given by*

$$H_j f_\lambda(\mathbf{x}) = (a_\delta(\mathbf{x}))^{-1} e_j(D_{x_1}^2, \dots, D_{x_L}^2) a_\delta(\mathbf{x}) f_\lambda(\mathbf{x}),$$

with corresponding eigenvalues

$$h_j(\lambda) = e_j(\mu_1^2, \dots, \mu_L^2),$$

recalling that $\mu = \lambda + \delta$.

Proof. The proof that these Hamiltonians satisfy (4.3.1) is equivalent to the proof of

$$e_j(D_{x_1}^2, \dots, D_{x_L}^2) a_\mu(\mathbf{x}) = e_j(\mu_1^2, \dots, \mu_L^2) a_\mu(\mathbf{x}).$$

We set $v_i^j = x_i^{\mu_j} - x_i^{-\mu_j}$ and note that $D_{x_i}^2 v_i^j = \mu_j^2 v_i^j$. Writing a_μ as

$$a_\mu(x_1, \dots, x_L) = \sum_{\sigma \in S_L} (-1)^\sigma v_1^{\sigma_1} \dots v_L^{\sigma_L},$$

we then have

$$\begin{aligned} e_j(D_{x_1}^2, \dots, D_{x_L}^2) a_\mu &= \sum_{1 \leq r_1 < \dots < r_j \leq L} \left(\sum_{\sigma \in S_L} D_{x_{r_1}}^2 \dots D_{x_{r_j}}^2 (-1)^\sigma v_1^{\sigma_1} \dots v_L^{\sigma_L} \right) \\ &= \sum_{\sigma \in S_L} \left(\sum_{1 \leq r_1 < \dots < r_j \leq L} \mu_{\sigma_{r_1}}^2 \dots \mu_{\sigma_{r_j}}^2 \right) (-1)^\sigma v_1^{\sigma_1} \dots v_L^{\sigma_L}. \end{aligned}$$

Due to symmetry, the inner sum is independent of σ , and equal to $e_j(\mu_1^2, \dots, \mu_L^2)$. This can be taken out of the outer sum, which is equal to a_μ . \square

4.4 Differential equation for q_λ

In this section we describe the single variable polynomial $q_\lambda(z)$, and define the differential operator W_i that satisfies (4.1.2).

Definition 4.4.1. The polynomial q_λ is given by

$$q_\lambda(z) = \frac{\chi_\lambda(z, 1, \dots, 1)}{\chi_\lambda(1, \dots, 1)}.$$

It will be useful to introduce $\phi_\mu(z_1, \dots, z_k)$, defined as

$$\phi_\mu(z_1, \dots, z_k) = a_\mu^{(k+1)}(z_1, \dots, z_k)/a_\mu^{(1)},$$

and in particular

$$\phi_\mu(z) = \sum_{j=1}^L \frac{z^{\mu_j} - z^{-\mu_j}}{\mu_j \prod_{i \neq j} (\mu_j^2 - \mu_i^2)}, \quad (4.4.1)$$

where we have used the identity (4.2.6). Recalling that $\mu = \lambda + \delta$, we can write $q_\lambda(z)$ as

$$q_\lambda(z) = \phi_\mu(z)/\phi_\delta(z). \quad (4.4.2)$$

Lemma 4.4.1. *The polynomial $\phi_\mu(z)$ satisfies the differential equation*

$$\prod_{n=1}^L (D_z^2 - \mu_n^2) \phi_\mu(z) = 0. \quad (4.4.3)$$

Proof. By definition,

$$D_z^2 \phi_\mu(z) = \sum_{j=1}^L \mu_j^2 \frac{(z^{\mu_j} - z^{-\mu_j})}{\mu_j \prod_{i \neq j} (\mu_j^2 - \mu_i^2)},$$

so the n th term of the sum in ϕ_μ is reduced to 0 by the n th factor in the product. \square

Lemma 4.4.2. *The differential equation $W q_\lambda(z) = 0$ is satisfied when*

$$W = \prod_{n=1}^L (Z^2 - \mu_n^2),$$

where

$$Z = D_z + \frac{(Lz^2 + 2Lz - 2z + L)}{(z^2 - 1)}.$$

Proof. Equation (4.4.3) can be written as

$$\prod_{n=1}^L (D_z^2 - \mu_n^2) \phi_\delta(z) q_\lambda(z) = 0. \quad (4.4.4)$$

Using (4.2.7) and (4.2.8), we obtain

$$\phi_\delta(z) = \frac{(z+1)(z-1)^{2L-1}}{z^L(2L-1)!},$$

and applying the derivative,

$$\begin{aligned} D_z \phi_\delta(z) &= \frac{(z-1)^{2L-2}(Lz^2 + 2Lz - 2z + L)}{z^L(2L-1)!} \\ &= \phi_\delta(z) \frac{(Lz^2 + 2Lz - 2z + L)}{(z^2 - 1)}. \end{aligned}$$

Since D_z obeys the same product rule as the usual derivative, we have

$$\begin{aligned} D_z \phi_\delta(z) f(z) &= \phi_\delta(z) \left(D_z + \frac{(Lz^2 + 2Lz - 2z + L)}{(z^2 - 1)} \right) f(z) \\ &= \phi_\delta(z) Z f(z). \end{aligned}$$

Substituting this result back into (4.4.4) we obtain

$$0 = \prod_{n=1}^L (Z^2 - \mu_n^2) q_\lambda(z),$$

for $z \neq \pm 1$. □

4.5 Inverse separating operator

The inverse of the operator $\mathcal{S} = \mathcal{A}_1 \dots \mathcal{A}_L$ must satisfy

$$\mathcal{S}^{-1} \prod_{i=1}^L q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \dots, x_L)}{\chi_\lambda(1, \dots, 1)},$$

for all λ .

Proposition 4.5.1. \mathcal{S}^{-1} is given by the differential operator

$$(\mathcal{S}^{-1} f)(x_1, \dots, x_L) = (-1)^{\frac{L(L-1)}{2}} \phi_\delta(x_1, \dots, x_L)^{-1} K_L \prod_{i=1}^L \phi_\delta(x_i) f(x_1, \dots, x_L),$$

where

$$K_L = \det_L \left[D_{x_i}^{2(L-j)} \right].$$

Proof. Acting with the operator \mathcal{S}^{-1} on the product of $q_\lambda(x_i)$ results, using (4.4.2), in K_L acting on a product of $\phi_\mu(x_i)$. Taking one factor of this product into each row of the determinant K_L ,

$$\det_L \left[D_{x_i}^{2(L-j)} \phi_\mu(x_i) \right] = \det_L \left[\sum_{m=1}^L \frac{\mu_m^{2(L-j)+1} (x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m^2 \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right],$$

where we have used the explicit expression for $\phi_\mu(x_i)$ given in (4.4.1). This can be expressed as the product of two matrices,

$$\begin{aligned} \det_L \left[\sum_{m=1}^L \frac{\mu_m^{2(L-j)+1} (x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m^2 \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right] &= \det_L \left[\mu_i^{2(L-j)+1} \right] \det_L \left[\frac{x_i^{\mu_j} - x_i^{-\mu_j}}{\mu_j^2 \prod_{n \neq j} (\mu_j^2 - \mu_n^2)} \right] \\ &= \frac{\det_L \left[\mu_i^{2(L-j)+1} \right] \det_L \left[x_i^{\mu_j} - x_i^{-\mu_j} \right]}{\prod_{m=1}^L (\mu_m^2 \prod_{n \neq m} (\mu_m^2 - \mu_n^2))}. \end{aligned}$$

The first determinant in the numerator is just $a_\mu^{(1)}$, see (4.2.5), and the second is the definition of $a_\mu(x_1, \dots, x_L)$. Since the denominator is equal to $(-1)^{\frac{L(L-1)}{2}} (a_\mu^{(1)})^2$, we finally obtain

$$K_L \prod_{i=1}^L \phi_\delta(x_i) q_\lambda(x_i) = (-1)^{\frac{L(L-1)}{2}} \phi_\mu(x_1, \dots, x_L),$$

from which it immediately follows that \mathcal{S}^{-1} satisfies the required property. \square

We can also find the inverse of the operator $\mathcal{S}_k = \mathcal{A}_1 \dots \mathcal{A}_k$, which must satisfy

$$\mathcal{S}_k^{-1} \prod_{i=1}^k q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \dots, x_k, 1, \dots, 1)}{\chi_\lambda(1, \dots, 1)}.$$

This will be useful for calculating the asymptotics of χ_λ with all but a few of its arguments set to one.

Proposition 4.5.2. \mathcal{S}_k^{-1} is given by the differential operator

$$(\mathcal{S}_k^{-1} f)(x_1, \dots, x_k) = (-1)^{\frac{k(k-1)}{2}} \phi_\delta(x_1, \dots, x_k)^{-1} \det_k \left[D_{x_i}^{2(k-j)} \right] \prod_{i=1}^k \phi_\delta(x_i) f(x_1, \dots, x_k).$$

The power of -1 here has been corrected from that which appears in the published version of [21].

Proof. The proof of this is equivalent to the proof of

$$\det_k \left[D_{x_i}^{2(k-j)} \right] \prod_{i=1}^k \phi_\mu(x_i) = (-1)^{\frac{k(k-1)}{2}} \phi_\mu(x_1, \dots, x_k). \quad (4.5.1)$$

On the LHS, we insert one factor $\phi_m(x_i)$ into row i of the determinant, so that each element of the matrix becomes

$$D_{x_i}^{2(k-j)} \phi_\mu(x_i) = \sum_{m=1}^L \mu_m^{2(k-j)} \frac{(x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m \prod_{n \neq m} (\mu_m^2 - \mu_n^2)}.$$

Using the proof of Proposition 4.5.1 as a guide, we rewrite the RHS as

$$(-1)^{\frac{k(k-1)}{2}} \frac{a_\mu^{(1)} a_\mu^{(k+1)}(x_1, \dots, x_k)}{(a_\mu^{(1)})^2},$$

and use the product formula for the denominator while using the determinant formula for the numerator, to produce

$$\begin{aligned} & (-1)^{\frac{k(k-1)}{2}} \frac{\det_L \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{i \leq k} \\ [\mu_j^{2(L-i)+1}]_{i > k} \end{bmatrix} \det_L [\mu_i^{2(L-j)+1}]}{(\prod_m \mu_m \prod_{n > m} (\mu_m^2 - \mu_n^2))^2} \\ &= (-1)^{\frac{k(k-1)+L(L-1)}{2}} \det_L \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{i \leq k} \\ [\mu_j^{2(L-i)+1}]_{i > k} \end{bmatrix} \det_L \left[\frac{\mu_i^{2(L-j)+1}}{\mu_i^2 \prod_{n \neq i} (\mu_i^2 - \mu_n^2)} \right] \\ &= (-1)^{\frac{k(k-1)+L(L-1)}{2}} \det_L \begin{bmatrix} \left[\sum_{m=1}^L \frac{\mu_m^{2(L-j)} (x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right]_{i \leq k} \\ \left[\sum_{m=1}^L \frac{\mu_m^{4L-2(i+j)+2}}{\mu_m^2 \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right]_{i > k} \end{bmatrix}. \end{aligned} \quad (4.5.2)$$

The elements in rows k to L can be written, with $\eta = i + j$, as

$$\begin{aligned} & \frac{1}{\prod_{r < s} (\mu_r^2 - \mu_s^2)} \sum_{m=1}^L (-1)^{m-1} \mu_m^{4L-2\eta} \prod_{\substack{1 \leq r < s \leq L \\ r, s \neq m}} (\mu_r^2 - \mu_s^2) \\ &= \frac{\det_L \left[\begin{bmatrix} \mu_i^{4L-2\eta} \end{bmatrix}_{j=1} \quad \begin{bmatrix} \mu_i^{2(L-j)} \end{bmatrix}_{j \geq 2} \right]}{\det_L [\mu_i^{2(L-j)}]}, \end{aligned}$$

which is 0 when $\eta > L + 1$, and 1 when $\eta = L + 1$. This means that (4.5.2) can be

expressed as

$$(-1)^{\frac{k(k-1)+L(L-1)}{2}} \det_L \begin{bmatrix} [*]_{\substack{i \leq k \\ j \leq L-k}} & \left[\sum_{m=1}^L \frac{\mu_m^{2(L-j)} (x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right]_{\substack{i \leq k \\ j \geq L-k+1}} \\ \mathbf{A} & \mathbf{0}_{\substack{i \geq k+1 \\ j \geq L-k+1}} \end{bmatrix},$$

where ‘*’ is an entry that does not contribute to the determinant, and \mathbf{A} is a matrix with 1’s on the backwards diagonal, 0’s below and ‘*’s above. Then the expression can be reduced to

$$\det_k \left[\sum_{m=1}^L \frac{\mu_m^{2(k-j)} (x_i^{\mu_m} - x_i^{-\mu_m})}{\mu_m \prod_{n \neq m} (\mu_m^2 - \mu_n^2)} \right],$$

which is exactly the LHS of (4.5.1). \square

4.6 The integral operator Q_z

In this section we will construct the operator Q_z satisfying

$$[Q_z \chi_\lambda](x_1, \dots, x_L) = q_\lambda(z) \chi_\lambda(x_1, \dots, x_L). \quad (4.6.1)$$

In order to construct Q_z as an integral operator, we will need to define an appropriate domain of integration, which we will do first. The integration variables are t_1, \dots, t_{L-1} , y_1, \dots, y_L , and w . The integration variables interlace the x ’s as $x_i \leq y_i/t_i \leq x_{i+1}$, for $i = 1, \dots, L-1$.

Definition 4.6.1. For any Laurent series $f(t) = \sum_{m \in \mathbb{Z}} c_m t^m$ with no constant term, i.e., $c_0 = 0$, the domain P of the integral over t is defined as

$$\int_P \frac{dt}{t} f(t) := \int_0^1 \frac{dt}{t} \sum_{m>0} c_m t^m - \int_1^\infty \frac{dt}{t} \sum_{m<0} c_m t^m = \sum_{m \in \mathbb{Z}} \frac{c_m}{m}.$$

For $1 \leq i \leq L-1$, we also define

$$\iint_{\mathcal{D}_i} dY_i = \int_P \frac{dt_i}{t_i} \int_{t_i x_i}^{t_i x_{i+1}} \frac{dy_i}{y_i},$$

and we will need two domains for the integrals over y_L ,

$$x_L < \frac{y_L}{t_1 \dots t_{L-1}} < \infty, \quad \text{and} \quad 0 < \frac{y_L}{t_1 \dots t_{L-1}} < x_1.$$

With these definitions we can now write down explicitly the operator Q_z that satisfies (4.6.1).

Proposition 4.6.1. Q_z is given by

$$[Q_z f](x_1, \dots, x_L) = \frac{1}{\phi_\delta(z) a_\delta(x_1, \dots, x_L)} \int_1^z \frac{dw}{w} \int_{\mathbf{D}} d\mathbf{Y} a_\delta(y_1, \dots, y_L) f(y_1, \dots, y_L),$$

where

$$\begin{aligned} \int_{\mathbf{D}} d\mathbf{Y} &= \iint_{\mathcal{D}_1} dY_1 \cdots \iint_{\mathcal{D}_{L-1}} dY_{L-1} \left[\int_{t_1 \dots t_{L-1} x_L}^\infty dy_L \delta \left(y_L - w x_L \prod_{l=1}^{L-1} \frac{x_l t_l^2}{y_l} \right) \right. \\ &\quad \left. + \int_0^{t_1 \dots t_{L-1} x_1} dy_L \delta \left(y_L - \frac{x_L}{w} \prod_{l=1}^{L-1} \frac{x_l t_l^2}{y_l} \right) \right]. \end{aligned}$$

Proof. The LHS of (4.6.1) becomes

$$\frac{1}{\phi_\delta(z) a_\delta(x_1, \dots, x_L)} \int_1^z \frac{dw}{w} \int_{\mathbf{D}} d\mathbf{Y} a_\mu(y_1, \dots, y_L),$$

while the RHS can be written as

$$\frac{\phi_\mu(z) a_\mu(x_1, \dots, x_L)}{\phi_\delta(z) a_\delta(x_1, \dots, x_L)}.$$

It is therefore enough to prove that

$$\int_1^z \frac{dw}{w} \int_{\mathbf{D}} d\mathbf{Y} a_\mu(y_1, \dots, y_L) = \phi_\mu(z) a_\mu(x_1, \dots, x_L). \quad (4.6.2)$$

To do this, we first use the following three ingredients:

- i. We expand $a_\mu(\mathbf{y})$ along the bottom row, producing

$$\sum_{r=1}^L (-1)^{L+r} (y_L^{\mu_r} - y_L^{-\mu_r}) \det_{L-1} \left[y_i^{\mu_j} - y_i^{-\mu_j} \right]_{\substack{i \neq L \\ j \neq r}}. \quad (4.6.3)$$

For each term in this sum, the integrals over y_L can be performed easily, resulting in

$$y_L^{\mu_r} - y_L^{-\mu_r} \rightarrow (w^{\mu_r} + w^{-\mu_r}) \left[\left(x_L \prod_{l=1}^{L-1} \frac{x_l t_l^2}{y_l} \right)^{\mu_r} - \left(\frac{1}{x_L} \prod_{l=1}^{L-1} \frac{y_l}{x_l t_l^2} \right)^{\mu_r} \right]. \quad (4.6.4)$$

- ii. The integral over w becomes elementary:

$$\int_1^z \frac{dw}{w} (w^{\mu_r} + w^{-\mu_r}) = \frac{1}{\mu_r} (z^{\mu_r} - z^{-\mu_r}). \quad (4.6.5)$$

iii. Combining (4.6.3) and (4.6.4) we find the further simplification

$$\begin{aligned}
& \left[\left(x_L \prod_{l=1}^{L-1} \frac{x_l t_l^2}{y_l} \right)^{\mu_r} - \left(\frac{1}{x_L} \prod_{l=1}^{L-1} \frac{y_l}{x_l t_l^2} \right)^{\mu_r} \right] \det_{L-1} \left[y_i^{\mu_j} - y_i^{-\mu_j} \right]_{\substack{i \neq L \\ j \neq r}} \\
&= \prod_{l=1}^L x_l^{\mu_r} \det_{L-1} \left[t_i^{2\mu_r} (y_i^{\mu_j - \mu_r} - y_i^{-\mu_j - \mu_r}) \right]_{\substack{i \neq L \\ j \neq r}} \\
&\quad - \prod_{l=1}^L x_l^{-\mu_r} \det_{L-1} \left[t_i^{-2\mu_r} (y_i^{\mu_j + \mu_r} - y_i^{\mu_r - \mu_j}) \right]_{\substack{i \neq L \\ j \neq r}},
\end{aligned}$$

after which the remaining integrals over Y_i can be inserted into each row of the determinants, e.g.,

$$\begin{aligned}
& \iint_{\mathcal{D}_1} dY_1 \cdots \iint_{\mathcal{D}_{L-1}} dY_{L-1} \det_{L-1} \left[t_i^{2\mu_r} (y_i^{\mu_j - \mu_r} - y_i^{-\mu_j - \mu_r}) \right]_{\substack{i \neq L \\ j \neq r}} \\
&= \det_{L-1} \left[\iint_{\mathcal{D}_i} dY_i t_i^{2\mu_r} (y_i^{\mu_j - \mu_r} - y_i^{-\mu_j - \mu_r}) \right]_{\substack{i \neq L \\ j \neq r}} \\
&= \det_{L-1} \left[\frac{1}{\mu_j^2 - \mu_r^2} (x_{i+1}^{\mu_j - \mu_r} - x_i^{\mu_j - \mu_r} - x_{i+1}^{-\mu_j - \mu_r} + x_i^{-\mu_j - \mu_r}) \right]_{\substack{i \neq L \\ j \neq r}} \\
&= \frac{(-1)^{L-1}}{\prod_{j \neq r} (\mu_r^2 - \mu_j^2)} \det_{L-1} \left[x_{i+1}^{-\mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{-\mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{\substack{i \neq L \\ j \neq r}}.
\end{aligned}$$

Using these ingredients, we finally find the following expression for the LHS of (4.6.2),

$$\begin{aligned}
& \sum_{r=1}^L \frac{z^{\mu_r} - z^{-\mu_r}}{\mu_r \prod_{j \neq r} (\mu_r^2 - \mu_j^2)} \times \\
& \quad (-1)^{r-1} \left(\prod_{l=1}^L x_l^{\mu_r} \det_{L-1} \left[x_{i+1}^{-\mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{-\mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{\substack{i \neq L \\ j \neq r}} \right. \\
& \quad \left. - \prod_{l=1}^L x_l^{-\mu_r} \det_{L-1} \left[x_{i+1}^{\mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{\mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{\substack{i \neq L \\ j \neq r}} \right). \quad (4.6.6)
\end{aligned}$$

At this point it has become clear that if $(-1)^{r-1}$ times the difference of products is independent of r (so that we can factor it out of the sum), the remaining sum will be precisely equal to $\phi_\mu(z)$. Hence, it remains to show that for each r ,

$$\begin{aligned}
a_\mu(\mathbf{x}) &= (-1)^{r-1} \left(\prod_{l=1}^L x_l^{\mu_r} \det_{L-1} \left[x_{i+1}^{-\mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{-\mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{\substack{i \neq L \\ j \neq r}} \right. \\
&\quad \left. - \prod_{l=1}^L x_l^{-\mu_r} \det_{L-1} \left[x_{i+1}^{\mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{\mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{\substack{i \neq L \\ j \neq r}} \right).
\end{aligned}$$

This can be achieved by increasing the size of the matrices in the determinants by one row and one column, at $i = 1$ and $j = r$, cancelling the factor of $(-1)^{r-1}$. The new column has entries of 0 except at $i = 1$, which is 1, and the new row has entries $x_1^{-\mu_r}(x_1^{\mu_j} - x_1^{-\mu_j})$ for the first determinant and $x_1^{\mu_r}(x_1^{\mu_j} - x_1^{-\mu_j})$ for the second. We then use row reduction, adding the 1st row to the 2nd row, then the 2nd to the 3rd, and so on. For ease of display, we place the additional column on the end, producing an extra factor of $(-1)^{L-r}$ that will be cancelled at the very end when we permute the column back to $j = r$. Thus we obtain

$$(-1)^{L-r} \left(\prod_{l=1}^L x_l^{\mu_r} \det_L \begin{bmatrix} [x_i^{-\mu_r}(x_i^{\mu_j} - x_i^{-\mu_j})]_{j \neq r} & [1] \end{bmatrix} - \prod_{l=1}^L x_l^{-\mu_r} \det_L \begin{bmatrix} [x_i^{\mu_r}(x_i^{\mu_j} - x_i^{-\mu_j})]_{j \neq r} & [1] \end{bmatrix} \right),$$

where $[1]$ denotes a column of 1's. The factors outside the determinants are then inserted into their respective rows, resulting in

$$= (-1)^{L-r} \left(\det_L \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{j \neq r} & [x_i^{\mu_r}] \end{bmatrix} - \det_L \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{j \neq r} & [x_i^{-\mu_r}] \end{bmatrix} \right).$$

These two determinants can be combined by simply performing the subtraction in the last column. Permuting this column back to $j = r$, this finally results in

$$\det_L [x_i^{\mu_j} - x_i^{-\mu_j}] = a_\mu(x_1, \dots, x_L). \quad \square$$

4.7 The integral operator A_k

The next and final step is to obtain the operator \mathcal{A}_k , which satisfies

$$(\rho_{k-1} Q_z f(\mathbf{x}))|_{z=x_k} = \mathcal{A}_k \rho_k f(\mathbf{x}), \quad (4.7.1)$$

where

$$\rho_k f(x_1, \dots, x_L) = f(x_1, \dots, x_k, 1, \dots, 1).$$

Proposition 4.7.1. *Relation (4.7.1) is satisfied by*

$$\mathcal{A}_k = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \frac{1}{\phi_\delta(x_k) a_\delta^{(k)}(x_1, \dots, x_{k-1})} \int_1^{x_k} \frac{dw}{w} \int_{\mathbf{D}^{(\varepsilon)}} d\mathbf{Y}^{(\varepsilon)} a_\delta^{(k+1)}(y_1, \dots, y_k),$$

where

$$\begin{aligned} \int_{\mathbf{D}(\varepsilon)} d\mathbf{Y}^{(\varepsilon)} &= \iint_{\mathcal{D}_1} dY_1 \cdots \iint_{\mathcal{D}_{k-1}} dY_{k-1} \left[\int_0^{t_1 \dots t_{k-1} x_1} dy_k \delta \left(y_k - \frac{e^\varepsilon}{w} \prod_{l=1}^{k-1} \frac{t_l^2 x_l}{y_l} \right) \right. \\ &\quad \left. + \int_{t_1 \dots t_{k-1} e^\varepsilon}^\infty dy_k \delta \left(y_k - w e^\varepsilon \prod_{l=1}^{k-1} \frac{t_l^2 x_l}{y_l} \right) \right], \end{aligned}$$

and

$$\iint_{\mathcal{D}_{k-1}} dY_{k-1} = \int_P \frac{dt_{k-1}}{t_{k-1}} \int_{t_{k-1} x_{k-1}}^{t_{k-1} e^\varepsilon} \frac{dy_{k-1}}{y_{k-1}}.$$

We remind the reader that the domain of integration P is given in Definition 4.6.1.

Proof. The χ_λ form a linear basis of $\mathbb{C}[y_1^\pm, \dots, y_L^\pm]^{\mathcal{W}_B}$, so it suffices to show that (4.7.1) holds for $f = \chi_\lambda$, and the general case will follow by linearity. We thus want to prove that $\mathcal{A}_k \rho_k \chi_\lambda(\mathbf{x}) = q_\lambda(x_k) \rho_{k-1} \chi_\lambda(\mathbf{x})$, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_1^{x_k} \frac{dw}{w} \int_{\mathbf{D}(\varepsilon)} d\mathbf{Y}^{(\varepsilon)} a_\mu^{(k+1)}(y_1, \dots, y_k) = \phi_\mu(x_k) a_\mu^{(k)}(x_1, \dots, x_{k-1}), \quad (4.7.2)$$

using the fact that $a_\delta^{(k+1)} \rho_k \chi_\lambda = a_\mu^{(k+1)}$.

This proof is very similar to that of Proposition 4.6.1, but with a few added subtleties. The differences in the proof are outlined here, and we give the details in Appendix F.

As in step i. of the proof of Proposition 4.6.1, the determinant $a_\mu^{(k+1)}(x_1, \dots, x_k)$ is expanded along a row, but this time along the k th row instead of the last. Also, the integrals over y_k take the place of that over y_L in this step. The simplification in step iii. is also very similar. The products outside the determinant now run from 1 to $k-1$, and contain an extra factor of $e^{\pm \varepsilon \mu_r}$. The $k-1$ integrals over Y_1, \dots, Y_{k-1} are inserted into the first $k-1$ rows of each determinant.

After these integrals are performed, the sizes of the matrices are increased as before. This time we choose the entries of the new row proportional to $\sinh(\varepsilon \mu_j)$ so that the matrices simplify after row reduction. After this, the prefactors in front of the two determinants can be cancelled, and the determinants can be combined.

The final step is to approximate to first order in ε , and then perform row reduction on the rows from k to L , in order to remove the dependence on μ_r from each row. The result is a factor $a_\mu^{(k)}(x_1, \dots, x_{k-1})$, which can be factored out of the remaining sum over r . This last sum is simply equal to $\phi_\mu(x_k)$, and the proof is complete. \square

Conclusion

Two physical quantities of interest for the Temperley–Lieb loop model at the special point $n = 1$ have been calculated in this thesis, the groundstate eigenvector and its normalisation in Chapter 2 and the boundary-to-boundary correlation function in Chapter 3. In addition, a key step has been laid down for the asymptotic analysis of these results in Chapter 4.

We have shown in Chapter 2 that the ground state eigenvector equation for the TL(1) loop model with open boundaries is equivalent to the q -deformed Knizhnik–Zamolodchikov equation. We have used this connection to obtain an explicit description of key components of the ground state $|\Psi\rangle$ for finite system sizes and without resorting to the Bethe ansatz. The derivation of the normalisation Z of the TL(1) ground state is presented in Section 2.4.4, with the result expressed as a product of four symplectic characters χ (also known as Schur polynomials of the type C root system). Our results for two open boundaries contain as special cases those of reflecting and mixed boundary conditions, which have been considered before [23, 27, 109]. The general result for the open boundary case has proved to be less tractable than those corresponding to these other boundary conditions, as it is not possible to express the eigenvector components as factorised operators acting on a ‘highest weight’ component. Though we were able to find symmetries and relations that the eigenvector components must satisfy, in some cases the relations did not contain enough information to specify the components exactly.

Our result for the normalisation leads in Chapter 3 to an explicit expression for a finite size correlation function F of the open TL(1) loop model. In a two-dimensional percolation model on a finite width lattice, this quantity is the average number of percolating clusters passing in between two chosen points. This observable also falls within a class of parafermionic correlation functions that can be shown to be discretely holomorphic, and hence are precursors to analytic correlators in the conformally invariant continuum limit. We stress furthermore that our result is exact for systems of finite width, and not asymptotic, which is unusual for systems that are not free fermionic. Our main results (3.5.1) and (3.5.2) are expressed in terms of the same symplectic character χ as appeared in the normalisation. Analogous results have been obtained for cylindrical boundary conditions for both site and bond percolation [69].

Naturally we desire to study the asymptotic limit of these results, as this limit corresponds to the thermodynamic limit of the system. In the case of the correlation function, we wish to use the asymptotic analysis to check our results against the continuum limit, which was calculated non-rigorously using conformal field theory by Cardy [12]. To calculate the asymptotic limit of Z and F , an asymptotic analysis of the symplectic character χ must first be conducted. We are particularly interested in the role the two boundaries play in the limit as L goes to infinity, so we would especially like to study the asymptotics of the symplectic character when all z_i are set to 1 but ζ_0 and ζ_L are kept general.

In Chapter 4 we have set out the Q -operator method of separation of variables for the symplectic character, as initiated by Sklyanin [92] and as applied to type A Jack polynomials in [56, 59]. The steps of the separation method are laid out in Section 4.1. The separation of variables for the symplectic character follows a process similar to that of the type A Schur polynomial (more commonly known as the Schur function). However, some of the intermediate steps are technically much more involved. In particular, the Q - and \mathcal{A} -operators contain a double integration in each variable, whereas the corresponding operators for the Schur function contain only one. This is related to the fact that the Hamiltonian for the symplectic character contains double derivatives.

The problem of asymptotics for the symplectic character can now be regarded as a problem of asymptotics for the separated polynomial, which is a product of single-variable polynomials q_λ . As mentioned earlier, we are particularly interested in the asymptotic limit of χ_λ when all but k variables are set to 1, and the operator \mathcal{S}_k of Proposition 4.5.2 and its inverse are useful here. The differential equation satisfied by q_λ , given in Section 4.4, is expected to be helpful in determining the asymptotics of q_λ . This is certainly worthy of further investigation.

Outlook

The XXZ quantum spin chain with open boundaries at both ends is closely related to the TL(1) loop model at the anisotropy point $\Delta = -1/2$, as discussed in the Introduction. We hope and expect that the results in this thesis will lead to the calculation of finite size correlation functions for the XXZ spin chain. The symplectic character can also be interpreted as a Toda lattice wave function, an observation which hints at an interesting link between the quantum integrable TL(1) model and classical integrable models. Other incarnations of the TL(1) model to which our results may be applied include the conformally invariant stochastic raise and peel model [78,81] and supersymmetric lattice models with boundaries [19,107].

In the case of the stochastic raise and peel model, the ground state is a stationary state distribution. In this setting an expression for the normalisation is important, so that the stationary state is a properly normalised probability distribution.

One way forward would be to express certain linear combinations of the components of the ground state eigenvector in terms of multiple contour integrals such as was done for reflecting boundary conditions [31,110]. However, the success of this approach seems to be related to the existence of factorised expressions for the ground state components, which exist for reflecting and mixed boundary conditions [23]. In the case of open boundaries, the lack of a pseudovacuum leads to a lack of such convenient factorised expressions, so we anticipate some fundamental difficulties with this approach. We further hope to make a connection between our solutions and Macdonald–Koornwinder polynomials of type (C^\vee, C) for specialised parameters [51], as well as with those in the form of Jackson integrals for q KZ equations on tensor product spaces, see [65,99] for the case of type A .

There is another, independent reason for computing the normalisation, which is in the context of the Razumov–Stroganov conjecture [3,17,84]. The original version of this conjecture was proved in [11], but the more general versions remain unproved. This conjecture states that there is an intriguing relation between the TL(1) ground state $|\Psi_L\rangle$ and the combinatorics of a fully packed loop (FPL) model on finite geometries, as well as other combinatorial objects such as alternating sign matrices and symmetric plane partitions [9]. In particular it states that the ground state normalisation is equal to the statistical mechanical partition function of an FPL model on a finite patch of

the square lattice. Such a link has been made for the ground state of the model with *identified* open boundaries, see [24], but for the model considered here, which has two genuinely open boundaries, it is not known which FPL geometry (if any) gives rise to a partition function equal to the normalisation of $|\Psi_L\rangle$ as given in (2.4.9). Understanding the underlying combinatorics for the general case of two boundaries will therefore lead to a deepening of our understanding of the Razumov–Stroganov correspondence, as well as to possible generalisations of symmetric plane partitions and alternating sign matrices.

The method of separation of variables for the symplectic character is a small part of a much larger theory of polynomials. An obvious extension of this work is to generalise the method to Jack polynomials of type C and BC , or even Koornwinder polynomials. It is not expected that the inverse separating operator will be easy to construct in the more general case, as it is still an unsolved problem in the case of type A Jack polynomials. However, another method for reversing the SoV process was used in [56], using a so-called lifting operator, and it is hoped that an analogue of this will prove useful for the more general C and/or BC polynomials.

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Appendix A

The Hecke algebra

R -matrices are fundamental objects in the study of integrable lattice models [2]. In a lattice model of width L , \check{R}_i describes the possible states at site i . The R -matrix defined in (1.2.1) is just one example of a more general R -matrix,

$$\check{R}_i(z) = a(z) + b(z)T_i, \quad 1 \leq i \leq L-1,$$

where the T_i are algebraic objects, and $a(z)$ and $b(z)$ are coefficient functions. The T_i satisfy certain relations such that when $a(z)$ and $b(z)$ are chosen appropriately, the Yang–Baxter equation,

$$\check{R}_i(z)\check{R}_{i+1}(zw)\check{R}_i(w) = \check{R}_{i+1}(w)\check{R}_i(zw)\check{R}_{i+1}(z),$$

is satisfied.

If the model in question is periodic, the site $i = L$ is identified with $i = 1$, and the Yang–Baxter equation is also valid for \check{R}_{L-1} and \check{R}_1 .

If the model has non-diagonal boundary conditions, K -matrices are introduced to describe the possible states at one or both of the boundaries. A more general form of a K -matrix is

$$\check{K}_i(z) = c(z) + d(z)T_i, \quad i = 0, L.$$

Again, $c(z)$ and $d(z)$ are coefficient functions and the algebraic object T_i is chosen so that the reflection equation,

$$\begin{aligned} \check{K}_0(z)\check{R}_1(zw)\check{K}_0(w)\check{R}_1(w/z) &= \check{R}_1(w/z)\check{K}_0(w)\check{R}_1(zw)\check{K}_0(z), & \text{for } i = 0, \\ \check{K}_L(z)\check{R}_{L-1}(zw)\check{K}_L(w)\check{R}_{L-1}(w/z) &= \check{R}_{L-1}(w/z)\check{K}_L(w)\check{R}_{L-1}(zw)\check{K}_L(z), & \text{for } i = L, \end{aligned}$$

is satisfied.

The relations satisfied by the T_i are those of the Hecke algebra [63, 89]. The Hecke algebra is related to the symmetric group, and it has a number of versions relating to the

Dynkin diagram ‘type’ classifications.

Definition A.1. The Hecke algebra of type A_L is generated by $\{T_i \mid 1 \leq i \leq L-1\}$, and defined by the relations

$$(T_i - q)(T_i + q^{-1}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

with the T_i commuting otherwise. The latter of these relations is commonly known as the braid relation.

The most relevant presentation for the Temperley–Lieb loop model is given by the simple transformation

$$e_i = T_i - q, \tag{A.1}$$

so that

$$e_i^2 = -(q + q^{-1})e_i, \quad e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}.$$

By setting $e_i e_{i+1} e_i - e_i = 0$, we obtain the sl_2 quotient known as the (zero-boundary) Temperley–Lieb algebra given in Definition 1.1.1, with $q = e^{i\gamma}$. A representation of the Temperley–Lieb algebra on the tensor product space $\otimes_{i=1}^L \mathbb{C}^2$ is

$$e_i \mapsto 1^{\otimes i-1} \otimes \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes 1^{\otimes L-i-1},$$

and is used to build the R -matrix of the six vertex model, \check{R}_{6v} .

By taking sl_k quotients of the Hecke algebra, one can build modules of rank $k-1$. Such higher rank modules are discussed in [32, 91]. For example, one can construct a representation on the tensor product space $\otimes_{i=1}^L \mathbb{C}^k$, which can be used to build an R -matrix generalising \check{R}_{6v} [6, 54].

In addition, systems with boundaries can be dealt with by including additional generators in the algebra, with boundary-type braid relations. We give one example here.

Definition A.2. The Hecke algebra of type B_L has the generators and relations of Definition A.1, along with the additional generator T_0 that satisfies

$$(T_0 - e^{i\omega_0})(T_0 - e^{-i\omega_0}) = 0, \quad T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0.$$

The presentation defined by (A.1) can be extended to the type B case by defining

$$e_0 = \frac{T_0 - e^{i\omega_0}}{2i \sin(\omega_0 + \gamma)},$$

so that

$$e_0^2 = \frac{-\sin \omega_0}{\sin(\omega_0 + \gamma)} e_0, \quad e_0 e_1 e_0 e_1 - e_0 e_1 = e_1 e_0 e_1 e_0 - e_1 e_0.$$

Again, the sl_2 quotient is obtained by setting both sides of this last relation to 0, and this produces the one-boundary Temperley–Lieb algebra from Definition 1.1.3.

Similarly, the Hecke algebra of type BC_L is defined by the generators and relations of type B_L , along with the additional generator T_L , which satisfies similar relations to T_0 , with T_{L-1} taking the place of T_1 . However, the type BC_L algebra is no longer finite-dimensional, as elements of the form $(T_0 T_1 \dots T_L)^n$ cannot be reduced. The presentation defined in (A.1) can again be extended, and the sl_2 quotient produces the two-boundary Temperley–Lieb algebra from Definition 1.1.4 [18].

Finally, it is worthwhile to point out that the operators a_i and s_i defined in (2.1.3) and (2.1.4) are generators of the Hecke algebra of type BC [23]. They are related to the T_i by

$$\begin{aligned} a_i &= q - T_i, & s_i &= q^{-1} + T_i, & 1 \leq i \leq L-1, \\ a_k &= \frac{e^{i\omega_k} - T_k}{2i \sin(\omega_k + \gamma)}, & s_k &= \frac{T_k - e^{-i\omega_k}}{2i \sin(\omega_k + \gamma)}, & k = 0, L, \end{aligned}$$

and satisfy

$$a_i s_i = s_i a_i = 0, \quad 0 \leq i \leq L,$$

along with the relations (we list them here for the a_i ; the relations satisfied by the s_i are identical)

$$\begin{aligned} a_i^2 &= (q + q^{-1}) a_i, & 1 \leq i \leq L-1, \\ a_k^2 &= \frac{\sin \omega_k}{\sin(\omega_k + \gamma)} a_k, & k = 0, L, \end{aligned}$$

and

$$\begin{aligned} a_i a_{i+1} a_i - a_i &= a_{i+1} a_i a_{i+1} - a_{i+1}, & 1 \leq i \leq L-1, \\ a_0 a_1 a_0 a_1 - a_0 a_1 &= a_1 a_0 a_1 a_0 - a_1 a_0, \\ a_L a_{L-1} a_L a_{L-1} - a_L a_{L-1} &= a_{L-1} a_L a_{L-1} a_L - a_{L-1} a_L. \end{aligned}$$

For the zero-boundary and one-boundary q KZ equations (corresponding to the type A and type B Hecke algebras respectively), the highest weight element of the solution vector can be found explicitly by fixing its degree and considering its symmetries. All the remaining elements can then be written in terms of operators acting on the highest weight element. These operators are Baxterised versions of the Hecke algebra generators

s_i [23], and have the form

$$h(u) = s_i - \frac{q^{u-1} - q^{1-u}}{q^u - q^{-u}}.$$

Because the type BC Hecke algebra is infinite-dimensional and lacks a highest weight element, this approach only partially works for the q KZ equation with two boundaries.

Appendix B

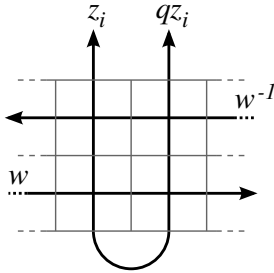
Proofs of the transfer matrix recursions

B.1 Proof of Proposition 2.2.1: The bulk recursion

In the following we will use the shorthand notation

$$a(z) = \frac{[q/z]}{[qz]}, \quad b(z) = -\frac{[z]}{[qz]}.$$

We first define $\alpha'_L = \varphi_i \alpha_{L-2}$ to be the link pattern of length L with a small link connecting sites i and $i+1$ inserted into the link pattern α_{L-2} . Restricting our focus to the action of the transfer matrix on the sites i and $i+1$, we have

$$T_L(w; z_{i+1} = qz_i) |\alpha'_L\rangle =$$


As each R -operator consists of two terms, the action of T_L on sites i and $i + 1$ produces sixteen terms,

$$\begin{aligned}
& a(z_i w) a(q z_i w) a(w/z_i) a(w/q z_i) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \\
& a(z_i w) a(q z_i w) a(w/z_i) b(w/q z_i) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots
\end{aligned}$$

Some of these pictures are equivalent with respect to their external connectivities. In total there are five different kinds of connectivities. For instance, one of the connectivities has

$$\begin{aligned}
& a(z_i w) b(q z_i w) a(w/z_i) b(w/q z_i) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \\
& a(z_i w) b(q z_i w) b(w/z_i) b(w/q z_i) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \\
& a(z_i w) b(q z_i w) a(w/z_i) a(w/q z_i) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} .
\end{aligned}$$

The closed loop in the first diagram is erased at the expense of a factor $-(q + q^{-1})$, after which the coefficients of the three diagrams sum to 0. Using the fact that $a(qu)a(u) + b(qu)b(u) - (q + q^{-1})a(qu)b(u) = 0$, it is easy to show that this happens for four of the

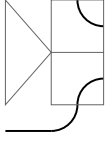
$$\begin{aligned}
& T_L(w; z_{i+1} = qz_i) | \alpha'_L \rangle \\
&= a(z_i w) b(qz_i w) b(w/z_i) a(w/qz_i) \\
&= a(z_i w) b(qz_i w) b(w/z_i) a(w/qz_i) \varphi_i \\
&= \frac{[q/z_i w][q^2 z_i/w]}{[q^2 z_i w][q w/z_i]} \varphi_i T_{L-2}(w; \hat{z}_i, \hat{z}_{i+1}) | \alpha_{L-2} \rangle .
\end{aligned}$$

Here we will describe the recursion of the transfer matrix at the left boundary. The recursion at the right boundary is similar.

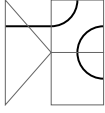
Setting $z_1 = q\zeta_0$, the entries of the eigenvector are zero unless they correspond to link patterns with a little link from position 1 to the left boundary. We have the transfer matrix acting on this little link,

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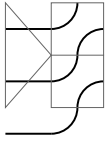
their weights:



$$\begin{aligned}
& \frac{[q/q\zeta_0 w]}{[q^2\zeta_0 w]} \left(\frac{[q^2\zeta_0/w]}{[qw/q\zeta_0]} + \frac{[q\zeta_0/w]}{[qw/q\zeta_0]} \right) \left(\frac{k(q/w, \zeta_0)}{k(w/q, \zeta_0)} + \frac{[1/q][q^2/w^2]}{k(w/q, \zeta_0)} \right) \\
& + \frac{[1/q\zeta_0 w]}{[q^2\zeta_0 w]} \frac{[q^2\zeta_0/w]}{[qw/q\zeta_0]} \frac{k(q/w, \zeta_0)}{k(w/q, \zeta_0)} \\
& = \frac{[1/\zeta_0 w]}{[q^2\zeta_0 w]} + \frac{[1/q\zeta_0 w][q^2\zeta_0/w][qw\zeta_0/q][qw/q\zeta_0]}{[q^2\zeta_0 w][w/\zeta_0][q^2/w\zeta_0][q^2\zeta_0/w]} \\
& = \frac{[1/\zeta_0 w]}{[q^2\zeta_0 w]} + \frac{[w\zeta_0]}{[q^2\zeta_0 w]} = 0,
\end{aligned}$$



$$\begin{aligned}
& \frac{[1/q\zeta_0 w]}{[q^2\zeta_0 w]} \frac{[q\zeta_0/w]}{[qw/q\zeta_0]} \left(\frac{k(q/w, \zeta_0)}{k(w/q, \zeta_0)} + \frac{[1/q][q^2/w^2]}{k(w/q, \zeta_0)} \right) \\
& = \frac{[q\zeta_0/w][1/q\zeta_0 w]}{[w/\zeta_0][q^2\zeta_0 w]} \\
& = \frac{k(q/w, q\zeta_0)}{k(w/q, q\zeta_0)},
\end{aligned}$$



$$\begin{aligned}
& \frac{[1/q\zeta_0 w]}{[q^2\zeta_0 w]} \frac{[q^2\zeta_0/w]}{[qw/q\zeta_0]} \frac{[1/q][q^2/w^2]}{k(w/q, \zeta_0)} \\
& = \frac{[1/q\zeta_0 w][q^2\zeta_0/w][1/q][q^2/w^2]}{[q^2\zeta_0 w][w/\zeta_0][q^2\zeta_0/w][q^2/w\zeta_0]} \\
& = \frac{[1/q][q^2/w^2]}{[w/\zeta_0][q^2\zeta_0 w]} \\
& = \frac{[1/q][q^2/w^2]}{k(w/q, q\zeta_0)}.
\end{aligned}$$

The second and third of these can be seen as the two contributions to a left boundary K -matrix, with an inserted loop connecting to the left boundary. The weights agree, and the whole sum can therefore be written as $\varphi_0 \circ \check{K}_0(w, q\zeta_0)$, in a system of size $L - 1$. This implies

$$T_L(\zeta_0; z_1, \dots) \circ \varphi_0 \Big|_{z_1=q\zeta_0} = \varphi_0 \circ T_{L-1}(q\zeta_0; z_2, \dots).$$

Appendix C

Symmetries of Z and the proportionality factors p_i , r_0 , and r_L

Recall the ground state eigenvector normalisation,

$$Z_L = \langle \Psi | \Psi \rangle,$$

which is equal to the sum over the components of $|\Psi\rangle$. Acting with the left eigenvector of the transfer matrix, $\langle \Psi|$, on both sides of the q KZ equation (2.1.1), and using the fact that π_i commutes with $\langle \Psi|$,

$$\begin{aligned} \pi_i Z_L &= \langle \Psi | \check{R}_i(z_i/z_{i+1}) | \Psi \rangle \\ &= \sum_{\alpha} \psi_{\alpha} \langle \Psi | \check{R}_i(z_i/z_{i+1}) | \alpha \rangle \\ &= \sum_{\alpha} \psi_{\alpha} \left(\frac{[qz_{i+1}/z_i]}{[qz_i/z_{i+1}]} \langle \Psi | \alpha \rangle - \frac{[z_i/z_{i+1}]}{[qz_i/z_{i+1}]} \langle \Psi | e_i | \alpha \rangle \right). \end{aligned}$$

Since $e_i | \alpha \rangle = | \alpha' \rangle$ for some link pattern α' , and $\langle \Psi | \alpha \rangle = 1$ for all α , this becomes

$$\begin{aligned} \pi_i Z_L &= \sum_{\alpha} \psi_{\alpha} \left(\frac{[qz_{i+1}/z_i]}{[qz_i/z_{i+1}]} - \frac{[z_i/z_{i+1}]}{[qz_i/z_{i+1}]} \right) \\ &= \sum_{\alpha} \psi_{\alpha} = Z_L. \end{aligned}$$

Similar arguments can be made to show that $\pi_0 Z_L = Z_L$ and $\pi_L Z_L = Z_L$. We therefore know that Z_L remains unchanged under any permutation of the variables z_i . Recalling

that

$$\sum_{\alpha} \psi_{\varphi_i \alpha}^L(z_{i+1} = qz_i) \varphi_i |\alpha\rangle = p_i(z_i; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) \sum_{\alpha} \psi_{\alpha}^{L-2}(\hat{z}_i, \hat{z}_{i+1}) \varphi_i |\alpha\rangle,$$

we have

$$Z_L(z_{i+1} = qz_i) = p_i(z_i; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) Z_{L-2}(\hat{z}_i, \hat{z}_{i+1}). \quad (\text{C.1})$$

Since Z is symmetric in the z_i , p_i becomes p_j under the transformation $z_i \leftrightarrow z_j$. We henceforth drop the index i from p_i . Further, since the LHS is symmetric in everything except z_i , as is Z_{L-2} , $p(z_i; \dots)$ must also be symmetric in all its variables except for z_i .

Completely analogously we can derive boundary recursions for the normalisation, which are

$$\begin{aligned} Z_L(z_1 = q\zeta_0, \dots, z_L; \zeta_0, \zeta_L) &= r_0(z_2, \dots, z_L; \zeta_0) Z_{L-1}(z_2, \dots, z_L; q\zeta_0, \zeta_L), \\ Z_L(z_1, \dots, z_L = \zeta_L/q; \zeta_0, \zeta_L) &= r_L(z_1, \dots, z_{L-1}; \zeta_L) Z_{L-1}(z_1, \dots, z_{L-1}; \zeta_0, \zeta_L/q). \end{aligned} \quad (\text{C.2})$$

As before, this shows that the proportionality factors r_0 and r_L must be symmetric in all their variables except for ζ_0 and ζ_L respectively.

Appendix D

Solving the q KZ equation for $L = 3$

Here we will solve the q KZ equation (2.1.2) with $L = 3$. For this example we will take $q = e^{2\pi i/3}$ and $s^4 = 1$, but we will leave b generic. We use the notation

$$\begin{aligned} \psi_1 &= \psi_{(((} & \psi_5 &= \psi_{)((} \\ \psi_2 &= \psi_{(())} & \psi_6 &= \psi_{)())} \\ \psi_3 &= \psi_{()((} & \psi_7 &= \psi_{))((} \\ \psi_4 &= \psi_{()())} & \psi_8 &= \psi_{)())}, \end{aligned}$$

and we recall the definition of s_i in (2.1.5)

$$s_i = -1 - a_i.$$

Considering in turn each i and α , the q KZ equation,

$$\sum_{\alpha} \psi_{\alpha}(e_i |\alpha\rangle) = - \sum_{\alpha} (a_i \psi_{\alpha}) |\alpha\rangle,$$

implies the following system of 32 equations:

$$\left. \begin{aligned} a_0 \psi_{\alpha} &= 0 \\ s_0 \psi_{\alpha+4} &= \psi_{\alpha} \end{aligned} \right\} \quad \alpha = 1, \dots, 4 \quad (\text{D.1})$$

$$\begin{aligned} a_1 \psi_{\alpha} &= 0 & \alpha &= 1, 2, 5, \dots, 8 \\ s_1 \psi_3 &= \psi_1 + \psi_2 + b\psi_5 + \psi_7 \\ s_1 \psi_4 &= \psi_6 + \psi_8 \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned}
a_2\psi_\alpha &= 0 & \alpha &= 1, 3, 4, 5, 7, 8 \\
s_2\psi_6 &= \psi_4 + \psi_5 + \psi_7 + \psi_8 \\
s_2\psi_2 &= \psi_1 + \psi_3
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
a_3\psi_{2\alpha} &= 0 & \alpha &= 1, \dots, 4 \\
s_3\psi_{2\alpha-1} &= \psi_{2\alpha} & \alpha &= 1, 3 \\
s_3\psi_{2\alpha-1} &= b\psi_{2\alpha} & \alpha &= 2, 4
\end{aligned} \tag{D.4}$$

The relations where the action of the projector a_i is zero entails certain symmetry restrictions on the components. For instance,

$$\begin{aligned}
\psi_1 &= \prod_{1 \leq i < j \leq 3} k(z_j, z_i) \prod_{i=1}^3 k(z_i, \zeta_0) f_1(z_1, z_2, z_3), \\
\psi_8 &= \prod_{1 \leq i < j \leq 3} k(1/sz_i, 1/sz_j) \prod_{i=1}^3 k(1/sz_i, 1/s\zeta_L) f_8(sz_1, sz_2, sz_3),
\end{aligned}$$

where the $f_\alpha(z_1, z_2, z_3)$ are symmetric functions invariant under $z_i \rightarrow 1/z_i$.

Using the system of equations we may obtain the components ψ_α in terms of ψ_1 and ψ_8 . Assuming we know ψ_1 , we find $\psi_2 = s_3\psi_1$, then $\psi_3 = s_2\psi_2 - \psi_1$, and $\psi_4 = b^{-1}s_3\psi_3$. Then we can find ψ_8 from

$$\psi_8 = s_2\psi_6 - \psi_4 - \psi_5 - \psi_7, \tag{D.5}$$

by using $\psi_6 = s_1\psi_4 - \psi_8$ and applying s_3 on both sides of (D.5),

$$-\psi_8 = s_3s_2(s_1\psi_4 - \psi_8) + \psi_4 - \psi_6 - b\psi_8,$$

which implies

$$(b-1)\psi_8 = (s_3s_2s_1 - s_1 + 1)\psi_4.$$

The expressions for ψ_2, ψ_3, ψ_4 as well as ψ_8 can be neatly rewritten in a factorised form as in [23]. However, if $b = 1$, the component ψ_8 cannot be determined this way.

In a similar way, given ψ_8 we can find ψ_4 , then ψ_6 , and ψ_2 . Since we can express ψ_4 and ψ_2 in two different ways, these components have to satisfy certain consistency conditions. Now we can find the remaining two components,

$$\begin{aligned}
(b-1)\psi_5 &= s_1\psi_3 - \psi_1 - \psi_2 - s_2\psi_6 + \psi_4 + \psi_8, \\
\psi_7 &= s_1\psi_3 - \psi_1 - \psi_2 - b\psi_5.
\end{aligned}$$

Again, if $b = 1$, ψ_5 and ψ_7 cannot be found separately in this way. However, their sum can be determined. Assuming we find an expression for one of these components (say, by

solving $s_0\psi_5 = \psi_1$ for ψ_5) that satisfies the appropriate degree, the boundary recursions in Lemma 2.2.3, and the symmetries imposed by (D.2) and (D.3), the entire system can be shown to be consistent.

At $b = 1$, as L becomes larger there are more and more components like ψ_5 and ψ_7 , which cannot be calculated directly from the extremal components. However, as above, certain sums of these components can be found, and consistency conditions can be checked. Using the recursions discussed in Section 2.2.2, recursive conditions can also be put on these components.

Appendix E

Inductive proof of Lemma 4.2.1

Proof. This proof shows that $a_\mu^{(k)}(x_1, \dots, x_{k-1})$ is proportional to $a_\mu^{(k+1)}(x_1, \dots, x_k)$ in the limit as $x_k \rightarrow 1$, and relies on the proof for $k = L$ given in Section 4.2.

We take $\lim_{\varepsilon \rightarrow 0} a_\mu^{(k+1)}(x_1, \dots, x_{k-1}, e^\varepsilon)$. In the k th row of the determinant (given in (4.2.1)), the j th element is

$$e^{\varepsilon \mu_j} - e^{-\varepsilon \mu_j} = 2 \sinh(\varepsilon \mu_j) = 2 \sum_{n=0}^{\infty} \frac{(\varepsilon \mu_j)^{2n+1}}{(2n+1)!}.$$

We use row reduction with the rows below the k th to remove the terms up to and including $n = L - (k + 1)$. The remainder of the series is

$$2 \frac{(\varepsilon \mu_j)^{2(L-k)+1}}{(2(L-k)+1)!} + \text{h.o.t.},$$

so $a_\mu^{(k)}(x_1, \dots, x_{k-1}) = \lim_{\varepsilon \rightarrow 0} \frac{(2(L-k)+1)!}{2\varepsilon^{2(L-k)+1}} a_\mu^{(k+1)}(x_1, \dots, x_{k-1}, e^\varepsilon)$. Then by induction we have that

$$a_\mu^{(k)}(x_1, \dots, x_{k-1}) \propto \lim_{x_k, \dots, x_L \rightarrow 1} a_\mu(x_1, \dots, x_L). \quad \square$$

Appendix F

Proof of Proposition 4.7.1

Proof. The proof of equation (4.7.2) is similar to that of Proposition 4.6.1:

- i. The first step is to expand the determinant $a_\mu^{(k+1)}$ over the k th row, giving

$$\sum_{r=1}^L (-1)^{k+r} (y_k^{\mu_r} - y_k^{-\mu_r}) \det_{L-1} \begin{bmatrix} [y_i^{\mu_j} - y_i^{-\mu_j}]_{i < k} \\ [\mu_j^{2(L-i)+1}]_{i > k} \end{bmatrix}_{j \neq r}. \quad (\text{F.1})$$

We can perform the integrals over y_k in (4.7.2), and, as before, the dependence on w factors out.

- ii. The integral over w has the same evaluation as (4.6.5).
- iii. The remaining factor is combined with the determinant,

$$\begin{aligned} & \left[\left(e^\varepsilon \prod_{l=1}^{k-1} \frac{t_l^2 x_l}{y_l} \right)^{\mu_r} - \left(e^{-\varepsilon} \prod_{l=1}^{k-1} \frac{y_l}{t_l^2 x_l} \right)^{\mu_r} \right] \det_{L-1} \begin{bmatrix} [y_i^{\mu_j} - y_i^{-\mu_j}]_{i < k} \\ [\mu_j^{2(L-i)-1}]_{k \leq i < L} \end{bmatrix}_{j \neq r} \\ &= e^{\varepsilon \mu_r} \prod_{l=1}^{k-1} x_l^{\mu_r} \det_{L-1} \begin{bmatrix} [t_i^{2\mu_r} (y_i^{\mu_j - \mu_r} - y_i^{-\mu_j - \mu_r})]_{i < k} \\ [\mu_j^{2(L-i)-1}]_{k \leq i < L} \end{bmatrix}_{j \neq r} \\ & \quad - e^{-\varepsilon \mu_r} \prod_{l=1}^{k-1} x_l^{-\mu_r} \det_{L-1} \begin{bmatrix} [t_i^{-2\mu_r} (y_i^{\mu_j + \mu_r} - y_i^{-\mu_j + \mu_r})]_{i < k} \\ [\mu_j^{2(L-i)-1}]_{k \leq i < L} \end{bmatrix}_{j \neq r}. \end{aligned}$$

The $k-1$ integrals over Y_i in (4.7.2) can be inserted into the first $k-1$ rows of the

determinants, and evaluated as before. The two determinants then become

$$\begin{aligned}
& e^{\pm \varepsilon \mu_r} \prod_{l=1}^{k-1} x_l^{\pm \mu_r} \det_{L-1} \left[\begin{array}{c} \left[\frac{1}{\mu_j^2 - \mu_r^2} (x_{i+1}^{\mp \mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{\mp \mu_r} (x_i^{\mu_j} - x_i^{-\mu_j})) \right]_{i < k-1} \\ \frac{1}{\mu_j^2 - \mu_r^2} (2 e^{\mp \varepsilon \mu_r} \sinh(\varepsilon \mu_j) - x_{k-1}^{\mp \mu_r} (x_{k-1}^{\mu_j} - x_{k-1}^{-\mu_j})) \\ \left[\mu_j^{2(L-i)-1} \right]_{k-1 < i < L} \end{array} \right]_{j \neq r} \\
&= \frac{(-1)^{L-1} e^{\pm \varepsilon \mu_r} \prod_{l=1}^{k-1} x_l^{\pm \mu_r}}{\prod_{j \neq r} (\mu_r^2 - \mu_j^2)} \\
& \quad \times \det_{L-1} \left[\begin{array}{c} \left[x_{i+1}^{\mp \mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{\mp \mu_r} (x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{i < k-1} \\ 2 e^{\mp \varepsilon \mu_r} \sinh(\varepsilon \mu_j) - x_{k-1}^{\mp \mu_r} (x_{k-1}^{\mu_j} - x_{k-1}^{-\mu_j}) \\ \left[(\mu_j^2 - \mu_r^2) \mu_j^{2(L-i)-1} \right]_{k-1 < i < L} \end{array} \right]_{j \neq r}.
\end{aligned}$$

Again, as in (4.6.6), we will extract the ingredients needed for $\phi_\mu(x_k)$ and show that the rest is independent of r so that it can be factored out of the sum in (F.1).

As in Section 4.6, we increase the size of each matrix by adding a row at $i = L$ and a column at $j = r$, introducing a factor of $(-1)^{L+r}$. The new column has entries equal to 0 except at $i = L$, which equals 1, and the new row has entries of $2 e^{\mp \varepsilon \mu_r} \sinh(\varepsilon \mu_j)$. We depict column r at the end:

$$\begin{aligned}
& (-1)^{r+1} e^{\pm \varepsilon \mu_r} \prod_{l=1}^{k-1} x_l^{\pm \mu_r} \\
& \quad \times \det_L \left[\begin{array}{ccc} \left[(x_{i+1}^{\mp \mu_r} (x_{i+1}^{\mu_j} - x_{i+1}^{-\mu_j}) - x_i^{\mp \mu_r} (x_i^{\mu_j} - x_i^{-\mu_j})) \right]_{i < k-1, j \neq r} & \mathbf{0} \\ \left[2 e^{\mp \varepsilon \mu_r} \sinh(\varepsilon \mu_j) - x_{k-1}^{\mp \mu_r} (x_{k-1}^{\mu_j} - x_{k-1}^{-\mu_j}) \right]_{j \neq r} & 0 \\ \left[(\mu_j^2 - \mu_r^2) \mu_j^{2(L-i)-1} \right]_{k-1 < i < L, j \neq r} & \mathbf{0} \\ \left[2 e^{\mp \varepsilon \mu_r} \sinh(\varepsilon \mu_j) \right]_{j \neq r} & 1 \end{array} \right].
\end{aligned}$$

Now we use row reduction, subtracting the L th row from the k th and then adding the $(k-1)$ th to the k th, the $(k-2)$ th to the $(k-1)$ th, etc. We then multiply the first

k rows by $-x_i^{\pm\mu_r}$, and the L th by $e^{\pm\epsilon\mu_r}$,

$$(-1)^{k-1} \det_L \begin{bmatrix} \left[(x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{i < k, j \neq r} & \left[x_i^{\pm\mu_r} \right]_{i < k} \\ \left[(\mu_j^2 - \mu_r^2) \mu_j^{2(L-i)-1} \right]_{k \leq i < L, j \neq r} & \mathbf{0} \\ [2 \sinh(\epsilon \mu_j)]_{j \neq r} & e^{\pm\epsilon\mu_r} \end{bmatrix}.$$

At this point we are able to combine the two determinants by subtracting the r th column of the second from the r th column of the first, obtaining

$$\det_L \begin{bmatrix} \left[(x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{i < k, j \neq r} & \left[(x_i^{\mu_r} - x_i^{-\mu_r}) \right]_{i < k} \\ \left[(\mu_j^2 - \mu_r^2) \mu_j^{2(L-i)-1} \right]_{k \leq i < L, j \neq r} & \mathbf{0} \\ [2 \sinh(\epsilon \mu_j)]_{j \neq r} & 2 \sinh(\epsilon \mu_r) \end{bmatrix}.$$

Now we approximate to first order in ϵ , resulting in $2\epsilon\mu_j$ in the bottom row. We can factor out 2ϵ , and then use row reduction once again: To each row in turn from $i = L-1$ to $i = k$ we add μ_r^2 times the row below, and we are finally left with

$$2\epsilon \det_L \begin{bmatrix} \left[(x_i^{\mu_j} - x_i^{-\mu_j}) \right]_{i < k} \\ \left[\mu_j^{2(L-i)+1} \right]_{i \geq k} \end{bmatrix} = 2\epsilon a_\mu^{(k)}(x_1, \dots, x_{k-1}).$$

This then factors out of the sum in (F.1), which is equal to $\phi_\mu(x_k)$. Putting everything together, the factors of (-1) cancel out, as does the factor of 2ϵ , and we are left with

$$a_\mu^{(k)}(x_1, \dots, x_{k-1}) \phi_\mu(x_k),$$

which is the RHS of (4.7.2). □